# 2. Difference Equation Solution Technique

## Tutorial 6

Consider the following model of a closed economy. Y denotes output, C denotes consumption expenditure, and I denotes investment expenditure. The subscripts t and t-1 refer to the respective time periods.

$$Y_t = C_t + I_t \tag{6.1}$$

$$C_t = 200 + 0.75Y_{t-1} \tag{6.2}$$

$$I_t = 150 + 0.15Y_{t-1} \tag{6.3}$$

1. Condense the model into a difference equation involving output and comment on its properties.

2. Solve for equilibrium output (assume $Y_0 = 4000$) and comment on the time path of output.

## Solution

1) Substitute equations (6.2) and (6.3) in (6.1) to get the reduced form equation in terms of $Y_t$:

$$\Rightarrow Y_t = 200 + 0.75Y_{t-1} + 150 + 0.15Y_{t-1}$$

or

$$Y_t - 0.9Y_{t-1} = 350 \tag{6.4}$$

What we have is a first-order nonhomogeneous difference equation in terms of $Y_t$.

**Conjecture** *Suppose we are seeking the solution to the first-order difference equation* $Y_{t+1} - aY_t = c$ *where a and c are two constants. The general solution will consist of the sum of two components: a particular integral* $y_p$ *(representing the intertemporal equilibrium level of y), which is any solution of the complete nonhomogeneous equation*

(6.4), *and a complementary function* $y_c$ (*representing the deviation of the model from equilibrium*), *which is the general solution of the reduced equation*.

Since, the complementary function of a first-order difference equation $y_c$ can be expressed as

$$y_c = A_1 b^t$$

we can write the complementary function for this particular problem as:

$$y_c = A_1(0.9)^t$$

2) The intertemporal equilibrium or the particular solution for the model can be computed by setting the characteristic equation to equilibrium (where $^*$ denotes equilibrium values):

$$Y^* - 0.9Y^* = 350$$

$$y_p = Y^* = 3500$$

Thus the general solution for the model can be expressed as:

$$Y_t = y_c + y_p = A_1(0.9)^t + 3500 \qquad 6.5$$

At time t=0 we have:

$$\Rightarrow Y_0 = A_1(0.9)^0 + 3500$$

$$\Rightarrow 4000 = A_1(0.9)^0 + 3500$$

$$A_1 = 500$$

Substituting $A_1 = 500$ in (6.5) yields:

$$Y_t = 500(0.9)^t + 3500$$

Whether the equilibrium is dynamically stable is a question of whether or not the complementary function will tend to zero as $t \rightarrow \infty$. The value of $b$ is of crucial importance in this regard. On the other hand the arbitrary constant $A$ only produces a *scale effect* without changing the configuration of the time path. Since the characteristic root $b = 0.9$ or $|b| \prec 1$ the model is convergent. The model is therefore stable as it gradually converges towards 3500 without oscillation over time.

# Tutorial 7

Consider the following model (Prof. Paul Samuelson's (1944) classic multiplier-accelerator model) which seeks to explore the dynamic process of income determination. Y denotes output, C, I, and G denote consumption, investment, and government expenditure respectively. The subscript t and t-1 refers to the respective time periods.

$$Y_t = C_t + I_t + G_t \tag{7.1}$$

$$C_t = C_0 + cY_{t-1} \tag{7.2}$$

$$I_t = I_0 + w(C_t - C_{t-1}) \tag{7.3}$$

where $0 \prec c \prec 1$, $w \succ 0$, and $G_t = G_0$.

1. Condense the model into a difference equation involving output.

2. Solve for equilibrium output.

3. Comment on the time path of output if $w = 0.9$ and $c = 0.5$.

## Solution

1) In order to condense the model into a difference equation involving output, write down the model in equilibrium form and take deviation of the model from equilibrium (where $^*$ denotes equilibrium values).

$$Y^* = C^* + I^* + G_0 \tag{7.4}$$

$$C^* = C_0 + cY^* \tag{7.5}$$

$$I^* = I_0 + w(C^* - C^*) \tag{7.6}$$

Model in deviation from equilibrium (denoting $Y_t - Y^* = Y_t$ for notational convenience):

$$Y_t = C_t + I_t \tag{7.7}$$

$$C_t = cY_{t-1} \tag{7.8}$$

$$I_t = w(C_t - C_{t-1}) \tag{7.9}$$

**Definition** *The backward shift or lag operator is defined by* $LX_t = X_{t-1}$, $L^n X_t = X_{t-n}$ *for* $n = ..., -2, -1, 0, 1, 2, .....$ *Formally, the operator* $L^n$ *maps one sequence into another sequence.*

Thus, we can re-write (7.9) by using the lag operator as;

$$I_t = w(\triangle C_t) = w(1 - L)C_t \qquad 7.10$$

Similarly we can express (7.8) as;

$$C_t = cY_{t-1} = cLY_t \qquad 7.11$$

Substituting (7.10) and (7.11) in (7.7) yields;

$$Y_t = cLY_t + w(1 - L)cLY_t$$

Collecting terms in $Y_t$ yields (note we are including the $C$ which denotes the constants omitted earlier):

$$\Rightarrow Y_t[1 - cL - w(1 - L)cL] = C$$
$$\Rightarrow Y_t[1 - c(1 + w)L + (wc)L^2] = C$$

$$Y_t - c(1 + w)Y_{t-1} + (wc)Y_{t-2} = C \qquad 7.12$$

What we have is a second-order nonhomogeneous difference equation in terms of $Y_t$.

2) In order to solve for equilibrium output revert back to equations 7.4,7.5, and 7.6. From 7.6 we get:

$$I^* = I_0 \qquad 7.13$$

Substitute equation (7.5) in (7.4) to get;

$$\Rightarrow Y^* = C_0 + cY^* + I_0 + G_0$$

$$Y^* = \frac{1}{(1-c)}(C_0 + I_0 + G_0)$$

where $\frac{1}{(1-c)}$ is the Keynesian multiplier.

3) Substitute $w = 0.9$ and $c = 0.5$ in (7.12) to compute the time path of output.

$$Y_t - 0.5(1+0.9)Y_{t-1} + (0.9)(0.5)Y_{t-2} = C$$

**Conjecture** *Suppose we are seeking the solution to the first-order difference equation $Y_{t+1} - aY_t = c$ where $a$ and $c$ are two constants. The complementary function of a first-order difference equation $y_c$ can be expressed as $y_c = A_1 b^t$.*

**Conjecture** *Trying out a solution of the form $y_t = Ab^t$ on the second-order difference equation yields $\longrightarrow$ $Ab^{t+2} + a_1Ab^{t+1} + a_2Ab^t = 0$ or, after cancelling out the (nonzero) common factor $Ab^t$, we can express the higher-order difference equation as $\longrightarrow$ $b^2 + a_1 b + a_2 = 0$. This quadratic equation possesses the two characteristic roots $b_1$ and $b_2$.*

$$\Rightarrow b_1, b_2 = \frac{0.95 \pm \sqrt{(-0.95)^2 - 4(1)(0.45)}}{2} = \frac{0.95 \pm \sqrt{0.90 - 1.8}}{2}$$

Since $b^2 \prec 4ac$ we have complex or imaginary roots. In the case of a complex root we can write the solution as:

$$Y_t = Ab^t \cos(\theta t - \varepsilon)$$

where

$$\cos\theta = \frac{-\frac{1}{2}a}{\sqrt{b} = \lambda} = \frac{-\frac{1}{2}(-0.95)}{\sqrt{0.45}} = \frac{0.475}{0.671} \equiv 0.708$$

$$\theta = \cos^{-1}(0.708) = 44.93$$

Note: Here $a$ and $b$ denote $a_1$ and $a_2$ respectively in Chiang's (1984, pp.513) treatment of higher-order differential equations.

For cycles $\frac{2\pi}{\theta} = \frac{360^\circ}{\theta} = \frac{360^\circ}{44.93} \equiv 8$ quarters.

# Tutorial 8

Consider the following model of a closed economy. The system consists of four equations in four endogenous variables ($C_t$, $Y_t$, $T_t$, and $I_t$) and one exogenous variable $G_t = G_0$.

$$Y_t = C_t + I_t + G_t \qquad 8.1$$

$$C_t = c_0 + c_1(Y_{t-1} - T_{t-1}) \qquad 8.2$$

$$I_t = i_0 + i_1(Y_{t-1} - Y_{t-2}) \qquad 8.3$$

$$T_t = \tau Y_t \qquad 8.4$$

1. Condense the model into a difference equation involving output.

2. Solve for equilibrium output and comment on the time path of output if;

$$\begin{aligned} c_0 &= 100 \\ i_0 &= 200 \\ c_1 &= 0.5 \\ i_1 &= 0.5 \\ \tau &= 0.2 \\ G_0 &= 500 \end{aligned}$$

## Solution

1) Substituting equations (8.2), (8.3), and (8.4) in (8.1) yields:

$$\Rightarrow Y_t = c_0 + i_0 + G_0 + [c_1(1-\tau) + i_1]Y_{t-1} - i_1 Y_{t-2}$$

$$Y_t - [c_1(1-\tau) + i_1]Y_{t-1} + i_1 Y_{t-2} = c_0 + i_0 + G_0 \equiv C \qquad 8.5$$

where ‘C’ denotes all the constants collected together. Equation (8.5) is a second-order nonhomogenous difference equation in output.

2) The particular integral or the intertemporal equilibrium of the model is given by writing (8.5) - the characteristic equation in equilibrium form (where $^*$ denotes equilibrium values):

$$\Rightarrow Y^* - [c_1(1-\tau) + i_1]Y^* + i_1 Y^* = c_0 + i_0 + G_0 \equiv C$$

$$Y^* = \frac{c_0 + i_0 + G_0}{(1 - c_1(1-\tau))} = \frac{800}{0.6} = 1333.33$$

The complementary function is given by (after substituting the values):

$$\Rightarrow Y_t - [0.5(1-0.2) + 0.5]Y_{t-1} + 0.5Y_{t-2} = c_0 + i_0 + G_0 \equiv C$$

or

$$Y_t - 0.9Y_{t-1} + 0.5Y_{t-2} = C$$

As explained in the previous tutorial a second-order difference equation of this form can be expressed as a quadratic equation of the form ($ax^2 + bx + c$). Hence, the characteristic roots are:

$$\Rightarrow b_1, b_2 = \frac{0.9 \pm \sqrt{(-0.9)^2 - 4(1)(0.5)}}{2} = \frac{0.9 \pm \sqrt{0.81 - 2}}{2}$$

Since $b^2 \prec 4ac$ we have complex or imaginary roots. In the case of a complex root we can write the solution as:

$$Y_t = Ab^t \cos(\theta t - \varepsilon)$$

where

$$\cos\theta = \frac{-\frac{1}{2}a}{\sqrt{b}} = \frac{-\frac{1}{2}(-0.9)}{\sqrt{0.5}} = \frac{0.45}{0.707} \equiv 0.64$$
$$\theta = \cos^{-1}(0.64) = 50.21$$

Note: Here $a$ and $b$ denote $a_1$and $a_2$ respectively in Chiang's (1984, pp.513) treatment of higher-order differential equations.

For cycles $\frac{2\pi}{\theta} = \frac{360^\circ}{\theta} = \frac{360^\circ}{50.21} \equiv$7.2 quarters.

# Tutorial 9

Consider the following modified version of Sargent (1987(a)) model of a closed economy. The operator $E$ refers to an expected value based on the information available at time t-1. All other notations have their usual meaning.

$$Y_t = C_t + (I_t - I_{t-1}) \qquad \text{(market clearing condition)} \qquad 9.1$$

$$C_t = -\Psi P_t \qquad \text{(demand curve)} \qquad 9.2$$

$$I_t = \alpha(E_{t-1}P_{t+1} - P_t) \qquad \text{(inventory demand)} \qquad 9.3$$

$$Y_t = \gamma E_{t-1}P_t + \xi_t \qquad \text{(supply curve)} \qquad 9.4$$

where $\alpha$, $\gamma$, and $\Psi \succ 0$. $\xi_t$ represents the effects of exogenous variables on supply. Assume perfect foresight so that $E_{t-1}P_t = P_t$ for all t.

1. Condense the model into a difference equation involving $P_t$ and comment on its time path.

## Solution

1) Substitute equations (9.2), (9.3), and (9.4) in (9.1) to get the reduced form in terms of $P_t$:

$$\Rightarrow \gamma P_t + \xi_t = -\Psi P_t + [\alpha(P_{t+1} - P_t) - \alpha(P_t - P_{t-1})]$$

$$\Rightarrow \alpha(P_{t+1} - P_t) - \alpha(P_t - P_{t-1}) - \Psi P_t - \gamma P_t = \xi_t$$

$$\Rightarrow \alpha P_{t+1} - (2\alpha + \Psi + \gamma)P_t + \alpha P_{t-1} = \xi_t$$

Dividing throughout by $\alpha$ yields:

$$P_{t+1} - \left(2 + \frac{\Psi + \gamma}{\alpha}\right)P_t + P_{t-1} = \frac{\xi_t}{\alpha}$$

**Definition** *A forward lag operator is defined by* $L^{-1}X_t = X_{t+1}$, $L^{-n}X_t = X_{t+n}$ *for* $n = \ldots, -2, -1, 0, 1, 2, \ldots..$ *Formally, the operator* $L^n$ *maps one sequence into another sequence.*

$$L^{-1}P_t - \left(2 + \frac{\Psi + \gamma}{\alpha}\right)P_t + LP_t = \frac{\xi_t}{\alpha}$$

Collecting terms in $P_t$ and then multiplying throughout by $L$ yields:

$$\Rightarrow \left[1 - \left(2 + \frac{\Psi + \gamma}{\alpha}\right)L + L^2\right]P_t = \frac{\xi_{t-1}}{\alpha}$$

Let $\left(2 + \frac{\Psi+\gamma}{\alpha}\right) = \phi$. Then

$$[1 - \phi L + L^2]P_t = \frac{\xi_{t-1}}{\alpha} \qquad 9.5$$

We need to factor the polynomial $1 - \phi L + L^2$ as:

$$[1 - \phi L + L^2] = (1 - \lambda_1 L)(1 - \lambda_2 L) = \begin{pmatrix} 1 - \lambda_2 L - \\ \lambda_1 L + \lambda_1\lambda_2 L^2 \end{pmatrix} \qquad 9.6$$

$$\Rightarrow (1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2)$$

so that we need $\lambda_1 + \lambda_2 = \phi$, $\lambda_1\lambda_2 = 1$.

For the second equality ($\lambda_1\lambda_2 = 1$) to hold $\lambda_2$ has to be the inverse of $\lambda_1$ ($\lambda_2 = \frac{1}{\lambda_1}$).

Thus we can rewrite (9.6) as:

$$[1 - \phi L + L^2] = (1 - \lambda L)(1 - \lambda^{-1}L)$$

Similarly we rewrite (9.5) as:

$$P_t = \frac{\alpha^{-1}\xi_{t-1}}{[1-\phi L + L^2]} = \frac{\xi_{t-1}}{\alpha(1-\lambda L)(1-\lambda^{-1}L)}$$

Thus the general solution (characteristic equation) for this second-order nonhomogeneous difference equation can be expressed as:

$$P_t = \frac{\xi_{t-1}}{\alpha(1-\lambda L)(1-\lambda^{-1}L)} + A_1\lambda^t + A_2\left(\frac{1}{\lambda}\right)^t$$

where $A_1$ and $A_2$ are the arbitrary constants and $\lambda$ ($b_1$) and $\frac{1}{\lambda}$ ($b_2$) are the characteristic roots.

Note that

$\phi = \left(2 + \frac{\Psi+\gamma}{\alpha}\right) \equiv \lambda_1 + \lambda_2 \succ 2$ since $\alpha$, $\gamma$, and $\Psi \succ 0$.

It follows that one of our roots necessarily exceeds 1, the other necessarily is less than 1. Since, the dominant root $|\lambda| \succ 1$, the price level would follow an explosive nonoscillatory path.

# Tutorial 10

Consider the following illustration from Sargent (1987(a)). The operator $E$ refers to an expected value. All other notations have their usual meaning. Let $M_t$ be the natural logarithm of the money supply, $P_t$ the log of the price level and $E_tP_{t+1}$ the log of the price expected to prevail at time t+1 based on the information available at time t. The model is

$$M_t - P_t = \alpha(E_tP_{t+1} - P_t) \qquad 10.1$$

$$E_tP_{t+1} - P_t = \beta(P_t - P_{t-1}) \qquad 10.2$$

where $\alpha \prec 0$ and $\beta \succ 0$.

1. Condense the model into a difference equation involving the price level and determine the long-run equilibrium value of $P_t$ once we impose the stability condition $\left|\frac{\alpha\beta}{1+\alpha\beta}\right| \prec 1$?

2. What is the long-run equilibrium value of $P_t$ if we assume perfect foresight? What sort of terminal condition is necessary to rule out the occurrence of runaway inflation?

## Solution

1) Substitute equation (10.2) in (10.1) to get:

$$M_t - P_t = \alpha\beta\left(P_t - P_{t-1}\right) \qquad 10.3$$

$$\Rightarrow \alpha\beta P_t - \alpha\beta L P_t + P_t = M_t$$

$$\Rightarrow (1 + \alpha\beta - \alpha\beta L)P_t = M_t$$

Dividing throughout by $1 + \alpha\beta$ yields:

$$P_t - \left( \frac{\alpha\beta L}{1+\alpha\beta} \right) P_t \equiv P_t \left( 1 - \frac{\alpha\beta L}{1+\alpha\beta} \right) = \frac{M_t}{1+\alpha\beta} \qquad 10.4$$

Suppose for notational convenience we call,

$$\begin{aligned} P_t &= y_t \\ \frac{M_t}{1+\alpha\beta} &= x_t \\ \frac{\alpha\beta}{1+\alpha\beta} &= \lambda \end{aligned}$$

Then we can rewrite (10.4) as:

$$y_t = \left( \frac{1}{1-\lambda L} \right) x_t = \sum_{j=0}^{\infty} \lambda^j x_{t-j}$$

Note

$$\sum_{j=0}^{\infty} \lambda^j x_{t-j} = x_t + \lambda x_{t-1} + \lambda^2 x_{t-2} + ....$$

$$\sum_{j=0}^{\infty} \lambda^j x_{t-j} = x_t + \lambda L x_t + \lambda^2 L^2 x_t + ....$$

$$\sum_{j=0}^{\infty} \lambda^j x_{t-j} = x_t (1 + \lambda L + \lambda^2 L^2 + ....)$$

(**Summation of an infinite series**)

$$\sum_{j=0}^{\infty} \lambda^j x_{t-j} = x_t \left( \frac{1}{1-\lambda L} \right)$$

As a result we can write the general solution as:

$$\Rightarrow P_t = \frac{1}{1+\alpha\beta}\sum_{j=0}^{\infty}\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^j M_{t-j} + Ab^t$$

$$P_t = \frac{1}{1+\alpha\beta}\sum_{j=0}^{\infty}\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^j M_{t-j} + A\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^t \qquad 10.5$$

where *A* is the arbitrary constant and *b* is the characteristic root. Consequently, this is the general solution (characteristic equation) which describes the entire time path of the price level given the time path of *M*.

In order to arrive at the *particular solution* we must set the model to equilibrium and solve for $P^*$ which denotes equilibrium price level. So we can write (10.5) as:

$$P^* = \frac{1}{1+\alpha\beta}\sum_{j=0}^{\infty}\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^j M^* + A\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^t$$

Since $\left|\frac{\alpha\beta}{1+\alpha\beta}\right| \prec 1$, the second term in our particular solution tends to zero as time tends to infinity.

$$\lim_{t\to\infty}\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^t \to 0$$

Hence we can write our particular solution as

$$\Rightarrow P^* = \frac{M^*}{1+\alpha\beta}\left(1+\frac{\alpha\beta}{1+\alpha\beta}+\left(\frac{\alpha\beta}{1+\alpha\beta}\right)^2+\ldots\right)$$

$$P^* = \frac{M^*}{1+\alpha\beta}\left(\frac{1}{1-\frac{\alpha\beta}{1+\alpha\beta}}\right) = M^*$$

Thus, the long-run effect of a once-and-for-all jump in money supply is to drive the price level up by an equal amount (provided the above stability condition is met).

2) If we assume perfect foresight then the money demand function becomes:

$$M_t - P_t = \alpha\left(P_{t+1} - P_t\right)$$

$$\Rightarrow \alpha P_{t+1} - \alpha P_t + P_t = M_t$$
$$\Rightarrow [\alpha L^{-1} + (1 - \alpha)]P_t = M_t$$

Dividing throughout by $\alpha L^{-1}$ yields:

$$\Rightarrow \left[1 + \frac{1 - \alpha}{\alpha L^{-1}}\right]P_t = \frac{M_t}{\alpha L^{-1}}$$
$$\Rightarrow \left[1 - \frac{(\alpha - 1)}{\alpha L^{-1}}\right]P_t = \frac{M_{t-1}}{\alpha}$$

or

$$P_t = \frac{M_{t-1}}{\alpha} \times \frac{1}{\left[1 - \frac{(\alpha - 1)}{\alpha L^{-1}}\right]}$$

Thus we can write the general solution as:

$$P_t = \frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha - 1}{\alpha}\right)^j M_{t-j} + A\left(\frac{\alpha - 1}{\alpha}\right)^t \tag{10.6}$$

In order to arrive at the *particular solution* we must set the model to equilibrium and solve for $P^*$ which denotes equilibrium price level. So we can write (10.6) as:

$$P^* = \frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha - 1}{\alpha}\right)^j M^* + A\left(\frac{\alpha - 1}{\alpha}\right)^t$$

Note that

$$\frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^{j}M^{*} = M^{*}$$

$$P^{*}=M^{*}+A\left(\frac{\alpha-1}{\alpha}\right)^{t}$$

However, since $(\frac{\alpha-1}{\alpha}) \succ 1$ as $\alpha \prec 0$ the second term i.e., $A(\frac{\alpha-1}{\alpha})^{t}$ would be explosive. We would therefore require that $A = 0$ (terminal condition) in order to rule out a bubble.

# Tutorial 11

Consider the following model of a closed economy. $\pi$ denotes the inflation rate, $u$ denotes the unemployment rate, and $m$ is the growth of money stock. Treat $\alpha$ as exogenous. The superscript $e$ refers to an expected value. The subscripts $t$ and $t$+1 refers to the respective time periods.

$$\begin{aligned} \pi_t &= \alpha - \beta u_t + \gamma \pi_t^e & \qquad & \alpha, \beta \succ 0,\ 0 \prec \gamma \preceq 1 \\ \pi_{t+1}^e &= \pi_t^e + \lambda(\pi_t - \pi_t^e) & & 0 \prec \lambda \preceq 1 \\ u_{t+1} &= u_t - \delta(m - \pi_{t+1}) & & \delta \succ 0 \end{aligned}$$

1. Condense the model into a difference equation involving

(i) the inflation rate,
(ii) the unemployment rate.

2. Solve for
(i) the equilibrium inflation rate,
(ii) the unemployment rate.

3. If $\alpha = 20$, $\beta = 10$, $\gamma = \frac{1}{2}$, $\lambda = \frac{1}{3}$, $\delta = \frac{1}{2}$;
find the time path of the inflation rate.

4. Comment on the time path of the rate of inflation if $\gamma = \lambda = 1$.

## Solution

1) Expressing the model in equilibrium (where $^*$ denotes equilibrium value):

$$\pi^* = \alpha - \beta u^* + \gamma \pi^{e*} \tag{11.1}$$

$$\pi^{e*} = \pi^{e*} + \lambda(\pi^* - \pi^{e*}) \tag{11.2}$$

$$u^* = u^* + \delta(m - \pi^*) \tag{11.3}$$

Model in deviation from equilibrium (denoting $\pi_t - \pi^* = \pi_t$ for notational convenience):

$$\pi_t = -\beta u_t + \gamma \pi_t^e \tag{11.4}$$

$$\pi_{t+1}^e = \pi_t^e + \lambda(\pi_t - \pi_t^e) \tag{11.5}$$

$$u_{t+1} = u_t + \delta \pi_{t+1} \tag{11.6}$$

Using the lag operator we can simplify equations (11.5) and (11.6) as follows:

$$\Rightarrow\ L^{-1}\pi_t^e = \pi_t^e + \lambda\pi_t - \lambda\pi_t^e$$
$$\Rightarrow\ \left(L^{-1} - 1 + \lambda\right)\pi_t^e = \lambda\pi_t$$

$$\pi_t^e = \frac{\lambda\pi_t}{\left(L^{-1} - 1 + \lambda\right)} \equiv \frac{\lambda L \pi_t}{[1 - L(1-\lambda)]} \tag{11.7}$$

$$\Rightarrow\ L^{-1}u_t = u_t + \delta L^{-1}\pi_t$$
$$\Rightarrow\ (L^{-1} - 1)u_t = \delta L^{-1}\pi_t$$

$$u_t = \frac{\delta L^{-1}\pi_t}{(L^{-1} - 1)} \equiv \frac{\delta\pi_t}{(1 - L)} \tag{11.8}$$

Substituting (11.7) and (11.8) in (11.4) yields:

$$\Rightarrow \pi_t = -\beta\left(\frac{\delta\pi_t}{(1-L)}\right) + \gamma\left(\frac{\lambda L\pi_t}{[1 - L(1-\lambda)]}\right)$$

Collecting terms in $\pi_t$ yields:

$$\Rightarrow \pi_t\left[\begin{array}{c}(1+\beta\delta) + (-1+\lambda-1-\beta\delta+\beta\delta\lambda-\gamma\lambda)L + \\ (1-\lambda+\gamma\lambda)L^2\end{array}\right] = 0$$

$$(1+\beta\delta)\pi_t + (-2+\lambda-\beta\delta+\beta\delta\lambda-\gamma\lambda)\pi_{t-1} + (1-\lambda+\gamma\lambda)\pi_{t-2} = 0$$

Note that by writing down the model in deviation form we have successfully omitted the constants i.e., $\alpha$ and $m$. We can now add the constants to the second-order difference equation in $\pi$.

$$(1+\beta\delta)\pi_t + (-2+\lambda-\beta\delta+\beta\delta\lambda-\gamma\lambda)\pi_{t-1} + (1-\lambda+\gamma\lambda)\pi_{t-2} = C \qquad 11.9$$

where $C$ denotes the omitted constants.

ΘFor a second-order difference equation the solution for the complementary function can be written as $\pi_c = A_1(b_1)^t + A_2(b_2)^t$ where $A_1$ and $A_2$ are the arbitrary constants and $b_1$ and $b_2$ are the characteristic roots. As we have seen already, a second-order difference equation can be represented as a quadratic equation and its roots can be computed by applying the quadratic formula,

$$b_1, b_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

If $(b^2 \succ 4ac)$ we have real roots. On the otherhand if $(b^2 \prec 4ac)$ we have complex roots.

In the case of complex roots we can write the solution simply as

$$\pi_t = A\lambda^t \cos(\theta t - \varepsilon)$$

where $\cos\theta = \frac{-\frac{1}{2}a}{\sqrt{b}=\lambda}$.

where $\lambda$ is the dampening factor.

$$\theta = \cos^{-1}\left(\frac{-\frac{1}{2}a}{\sqrt{b} = \lambda}.\right)$$

In order to obtain a difference equation in terms of the unemployment rate go back to equation (11.8):

$$u_t = \frac{\delta\pi_t}{(1 - L)}$$

or

$$\pi_t = \frac{u_t(1 - L)}{\delta} \qquad 11.10$$

Substitute (11.10) in (11.9) for $\pi_t$, $\pi_{t-1}$ etc to obtain a second-order nonhomogenous difference equation in terms of the unemployment rate.

2(i) For equilibrium inflation rate go back to equation (11.3). Note that $m - \pi^* = 0$ that is, equilibrium inflation is equal to the growth in money supply.

2(ii) For equilibrium unemployment rate go to equation (11.2). From equation (11.2) we know that $\pi^{e*} = \pi^*$. Substituting this in equation (11.1) yields:

$$u^* = \frac{\alpha - (1 - \gamma)\pi^*}{\beta}$$

When $\gamma = 1$, we have $u^* = \frac{\alpha}{\beta}$. That is the long-run aggregate supply or the Phillips curve is vertical.

3. Substituting the given values in equation (11.9) (ignoring the constant term for simplicity) yields:

$$\left(1 + 10 \times \frac{1}{2}\right)\pi_t + \left(-2 + \frac{1}{3} - 10 \times \frac{1}{2} + 10 \times \frac{1}{2} \times \frac{1}{3} - \frac{1}{2} \times \frac{1}{3}\right)\pi_{t-1}$$

$$+ \left(1 - \frac{1}{3}\left(1 - \frac{1}{2}\right)\right)\pi_{t-2} = 0$$

or

$$6\pi_t - \frac{31}{6}\pi_{t-1} + \frac{5}{6}\pi_{t-2} = 0$$

Dividing throughout by 6 yields:

$$\pi_t - 0.86\pi_{t-1} + 0.14\pi_{t-2} = 0$$

Therefore the roots of this quadratic equation are:

$$b_1, b_2 = \frac{0.86 \pm \sqrt{0.74 - 0.56}}{2} \qquad b_1 = 0.22 \text{ and } b_2 = 0.64$$

Since, $(b^2 \succ 4ac)$ we have real roots. Thus the complementary function is given by $\pi_c = A_1(0.22)^t + A_2(0.64)^t$ where $b_2$ is the dominant root. As $t \rightarrow \infty$ this model converges in the form of a step function.

4. Substituting the given values in equation (11.9) (ignoring the constant term for simplicity) yields:

$$\Rightarrow 6\pi_t - 2\pi_{t-1} + \pi_{t-2} = 0$$

Dividing throughout by 6 yields:

$$\pi_t - 0.33\pi_{t-1} + 0.17\pi_{t-2} = 0$$

$$b_1, b_2 = \frac{0.33 \pm \sqrt{0.11 - 0.68}}{2}$$

Since, $(b^2 \prec 4ac)$ we have complex roots.

In the case of complex roots we can write the solution simply as:

$$\pi_t = A\lambda^t \cos(\theta t - \varepsilon)$$

where $\cos\theta = \frac{-\frac{1}{2}(-0.33)}{\sqrt{0.17}} = 0.42$

Note: Here $a$ and $b$ denote $a_1$and $a_2$ respectively in Chiang's (1984, pp.513) treatment of higher-order differential equations.

$$\theta = \cos^{-1}(0.42) \equiv 65.17$$

For cycles $\frac{360^\circ}{65.17} = 5.5$ quarters (provided the model is a quarterly model).

# Tutorial 12

Consider the following model of a closed economy. $p$ denotes the price level, $x$ denotes output, and $\overline{\Delta m}$ is the growth of money stock. The superscript $e$ refers to an expected value. The subscripts $t$ and $t$-1 refers to the respective time periods.

$$\begin{aligned}\Delta p_t &= ax_{t-1} + \Delta p^e_{t-1}\\ \Delta p^e_t &= \lambda\Delta p_t + (1-\lambda)\Delta p^e_{t-1}\\ \Delta x_t &= \delta(\overline{\Delta m} - \Delta p_t)\end{aligned}$$

1. Condense the model into a difference equation involving
(i) the price level,
(ii) full-employment output.

2. Solve for
(i) the equilibrium price level,
(ii) the equilibrium output.

3. If $a = 0.2$, $\lambda = 0.2$, $\delta = 0.1$;
find the time path of the price level.

4. Comment on the time path of the price level if $\lambda = 1$.

## Solution

1) Expressing the model in equilibrium (where $^*$ denotes equilibrium value);

$$\Delta p^* = ax^* + \Delta p^{e*} \tag{12.1}$$

$$\Delta p^{e*} = \lambda\Delta p^* + (1-\lambda)\Delta p^{e*} \tag{12.2}$$

$$\Delta x^* = \delta(\overline{\Delta m} - \Delta p^*) \tag{12.3}$$

Model in deviation from equilibrium (denoting $\pi_t - \pi^* = \pi_t$ for notational convenience):

$$\Delta p_t = ax_{t-1} + \Delta p^e_{t-1} \qquad 12.4$$

$$\Delta p^e_t = \lambda \Delta p_t + (1-\lambda)\Delta p^e_{t-1} \qquad 12.5$$

$$\Delta x_t = -\delta \Delta p_t \qquad 12.6$$

Using the lag operator we can simplify equations (12.5) and (12.6) as follows:

$$\Rightarrow (1-L)p^e_t = \lambda(1-L)p_t + (1-\lambda)Lp^e_t(1-L)$$

$$\Rightarrow p^e_t[1-(1-\lambda)L] = \lambda p_t$$

$$p^e_t = \frac{\lambda p_t}{[1-(1-\lambda)L]} \qquad 12.7$$

$$\Rightarrow (1-L)x_t = -\delta(1-L)p_t$$

$$x_t = -\delta p_t \qquad 12.8$$

Substituting (12.7) and (12.8) in (12.4) yields:

$$\Rightarrow (1-L)p_t = aL(-\delta p_t) + L\left(\frac{\lambda p_t}{[1-(1-\lambda)L]}\right)(1-L)$$

$$\begin{aligned}(1-L+\lambda L-L+L^2-\lambda L^2)p_t &= -a\delta p_t L + a\delta L^2 p_t - a\delta p_t \lambda L^2 \\ &\quad + L\lambda p_t - \lambda p_t L^2\end{aligned}$$

Collecting terms in $p_t$ yields:

$$\Rightarrow p_t\left[\begin{matrix}1-L+\lambda L-L+L^2-\lambda L^2+a\delta L-a\delta L^2+a\delta\lambda L^2- \\ L\lambda+\lambda L^2\end{matrix}\right] = C$$

$$p_t - [2-a\delta]p_{t-1} + [1-a\delta(1-\lambda)]p_{t-2} = C \qquad 12.9$$

where $C$ denotes the omitted constants.

In order to obtain a difference equation in terms of output go back to equation (12.8):

$$x_t = -\delta p_t$$

or

$$p_t = -\left(\frac{1}{\delta}\right)x_t \qquad 12.10$$

Substitute (12.10) in (12.9) for $p_t$, $p_{t-1}$ etc to obtain a second-order nonhomogenous difference equation in terms of output.

2) In order to compute equilibrium output go to equation (12.2);

$$\Delta p^{e*} = \lambda\Delta p^* + (1-\lambda)\Delta p^{e*}$$

$$\Rightarrow \Delta p^{e*}[1-(1-\lambda)] = \lambda\Delta p^*$$

$$\Delta p^{e*} = \Delta p^*$$

Substitute this in (12.1) to get equilibrium output:

$$x^* = 0$$

Substitute the value of equilibrium output in (12.3) to get equilibrium price level:

$$\overline{\Delta m} = \Delta p^*$$

3. Substituting the given values in equation (12.9) (ignoring the constant term for simplicity) yields:

$$\Rightarrow p_t - [2 - (0.2)(0.1)]p_{t-1} + [1 - (0.2)(0.1)(1 - 0.2)]p_{t-2} = 0$$

or

$$p_t - 1.98p_{t-1} + 0.984p_{t-2} = 0$$

Therefore the roots of this quadratic equation are;

$$b_1, b_2 = \frac{1.98 \pm \sqrt{3.92 - 3.94}}{2}$$

Since ($b^2 \prec 4ac$) we have complex roots.

In the case of complex roots we can write the solution simply as:

$$p_t = A\lambda^t \cos(\theta t - \varepsilon)$$

where $\cos\theta = \frac{-\frac{1}{2}(-1.98)}{\sqrt{0.984}} = 0.998$

$$\theta = \cos^{-1}(0.998) \equiv 3.62$$

For cycles $\frac{360^\circ}{3.62} = 99$ quarters (provided the model is a quarterly model).

4. Substituting the given values in equation (12.9) (ignoring the constant term for simplicity) yields:

$$p_t - 1.98p_{t-1} + p_{t-2} = 0$$

$$b_1, b_2 = \frac{1.98 \pm \sqrt{3.92 - 4}}{2}$$

Since ($b^2 \prec 4ac$) we have complex roots.

In the case of complex roots we can write the solution simply as

$$p_t = A\lambda^t \cos(\theta t - \varepsilon)$$

where $\cos\theta = \frac{-\frac{1}{2}(-1.98)}{\sqrt{1}} = 0.99$

$$\theta = \cos^{-1}(0.99) \equiv 8.11$$

For cycles $\frac{360^\circ}{8.11} = 44$ quarters (provided the model is a quarterly model).

# Tutorial 13

Consider the following Neo-Classical/Keynesian Synthesis model where notations have their usual meaning.

$$y_t = -\alpha r_t + \overline{d} \qquad \text{IS curve} \tag{13.1}$$

$$\overline{m} = p_t + \gamma y_t - \beta R_t \qquad \text{LM curve} \tag{13.2}$$

$$\dot{p}_t = \dot{p}^{e}_t + \delta(y_t - y^*) \qquad \text{Phillips curve} \tag{13.3}$$

$$\Delta \dot{p}^{e}_t = \lambda(\dot{p}_{t-1} - \dot{p}^{e}_{t-1}) \qquad \text{Adaptive expectations} \tag{13.4}$$

$$R_t = r_t + \dot{p}^{e}_{t+1} \qquad \text{Fisher equation} \tag{13.5}$$

where $\dot{x}_t = \Delta x_t = x_t - x_{t-1}$.

1.Condense the model into a difference equation involving the price level.

2. If $\alpha = 0.5$, $\beta = 3$, $\delta = 0.2$, $\gamma = 1$, and $\lambda = 0.1$; find the time path of the price level.

## Solution

1) Expressing the model in equilibrium (where $^*$ denotes equilibrium value):

$$y^* = -\alpha r^* + \overline{d} \tag{13.6}$$

$$\overline{m} = p^* + \gamma y^* - \beta R^* \tag{13.7}$$

$$\dot{p}^* = \dot{p}^{e*} + \delta(y^* - y^*) \tag{13.8}$$

$$\Delta \dot{p}^{e*} = \lambda\left(p^* - \dot{p}^{e*}\right) \tag{13.9}$$

$$R^* = r^* + \dot{p}^{e*} \tag{13.10}$$

Model in deviation from equilibrium (denoting $y_t - y^* = y_t$ for notational convenience):

$$y_t = -\alpha r_t \qquad 13.11$$

$$0 = p_t + \gamma y_t - \beta R_t \qquad 13.12$$

$$\dot{p}_t = \dot{p}_t^{e} + \delta y_t \qquad 13.13$$

$$\Delta \dot{p}_t^{e} = \lambda(\dot{p}_{t-1} - \dot{p}_{t-1}^{e}) \qquad 13.14$$

$$R_t = r_t + \dot{p}_{t+1}^{e} \qquad 13.15$$

Using the lag operator we can simplify equations (13.14) as follows:

$$(1-L)\dot{p}_t^{e} = \lambda L \dot{p}_t - \lambda L \dot{p}_t^{e}$$

$$\dot{p}_t^{e} = \frac{\lambda L \dot{p}_t}{(1 - L + \lambda L)} \equiv \frac{\lambda L \dot{p}_t}{(1-(1-\lambda)L)} \qquad 13.16$$

Substituting (13.16) in (13.13) yields:

$$\Rightarrow \dot{p}_t = \frac{\lambda L \dot{p}_t}{(1-(1-\lambda)L)} + \delta y_t$$

Multiplying throughout by $(1-(1-\lambda)L)$ yields:

$$\Rightarrow \dot{p}_t[1-L] = \delta y_t[1-(1-\lambda)L]$$

$$y_t = \frac{\dot{p}_t[1-L]}{\delta[1-(1-\lambda)L]} \qquad 13.17$$

From (13.11) we get:

$$r_t = -\frac{y_t}{\alpha} \qquad 13.18$$

Substituting (13.16) and (13.18) in (13.15) yields:

$$R_t = -\frac{y_t}{\alpha} + L^{-1}\left(\frac{\lambda L \dot{p}_t}{(1-(1-\lambda)L)}\right) \qquad 13.19$$

Substituting (13.19) and (13.17) in (13.12) yields:

$$0 = p_t + \gamma y_t - \beta\left(-\frac{y_t}{\alpha} + L^{-1}\left(\frac{\lambda L p_t^{\bullet}}{(1-(1-\lambda)L)}\right)\right)$$

$$0 = p_t + \left(\gamma + \frac{\beta}{\alpha}\right)y_t - \frac{\beta\lambda\, p_t^{\bullet}}{(1-(1-\lambda)L)}$$

$$0 = p_t + \left(\gamma + \frac{\beta}{\alpha}\right)\left(\frac{p_t^{\bullet}[1-L]}{\delta[1-(1-\lambda)L]}\right) - \frac{\beta\lambda\, p_t^{\bullet}}{(1-(1-\lambda)L)}$$

Note that $p_t^{\bullet} = p_t - p_{t-1} = (1-L)p_t$:

$$0 = p_t + \left(\gamma + \frac{\beta}{\alpha}\right)\left(\frac{[1-L]^2 p_t}{\delta[1-(1-\lambda)L]}\right) - \frac{\beta\lambda\,(1-L)p_t}{(1-(1-\lambda)L)}$$

Multiplying throughout by $[1-(1-\lambda)L]$ yields:

$$0 = p_t(1-(1-\lambda)L) + [1-L]^2\left(\frac{\gamma\alpha+\beta}{\alpha\delta}\right)p_t - p_t(1-L)\beta\lambda$$

Collecting terms in $p_t$ yields a second-order nonhomogenous difference equation.

$$\left(1 + \frac{\gamma\alpha+\beta}{\alpha\delta} - \beta\lambda\right)p_t + \left(-1 + \lambda - 2\left(\frac{\gamma\alpha+\beta}{\alpha\delta}\right) + \beta\lambda\right)p_{t-1}$$

$$+\left(\frac{\gamma\alpha+\beta}{\alpha\delta}\right)p_{t-2} = C$$

where $C$ denotes the omitted constants.

2) Substituting the given values above ( ignoring the constant term for simplicity) yields:

$$\left(1+\frac{1(0.5)+3}{0.5(0.2)}-3(0.1)\right)p_t+$$

$$\left(-1+0.1-2\left(\frac{1(0.5)+3}{0.5(0.2)}\right)+3(0.1)\right)p_{t-1}+$$

$$\left(\frac{1(0.5)+3}{0.5(0.2)}\right)p_{t-2}=0$$

$$\Rightarrow 35.7p_t-70.6p_{t-1}+35p_{t-2}=0$$

Dividing throughout by 35.7 yields:

$$p_t-1.978p_{t-1}+0.98p_{t-2}=0$$

Therefore the roots of this quadratic equation are:

$$b_1,b_2=\frac{1.978\pm\sqrt{3.912-3.92}}{2}$$

Since, ($b^2 \prec 4ac$) we have complex roots.

In the case of complex roots we can write the solution simply as

$$p_t=A\lambda^t\cos(\theta t-\varepsilon)$$

where $\cos\theta=\frac{-\frac{1}{2}(-1.978)}{\sqrt{0.98}}=0.999$

$$\theta=\cos^{-1}(0.999)\equiv 2.56$$

For cycles $\frac{360^\circ}{2.56}=141$ quarters (provided the model is a quarterly model).