3. Rational Expectations Solution Technique

Tutorial 14

1. Consider the following model of a closed economy where symbols have their usual meanings. *y*, *p* and *m* are in natural logarithms. *E* is the expectations operator and ε is a random disturbance term with the usual properties.

$$y_t = 20 + 0.25(p_t - E_{t-1}p_t) \qquad 14.1$$

$$y_t = 30 + 0.25(m_t - p_t) \qquad 14.2$$

$$m_t = 5 + \varepsilon_t \qquad 14.3$$

(a) Solve for

(i) the full-employment equilibrium value of y,
(ii) the expected price level.

(b) If $\varepsilon_t = -2$, what is (i) the price level? (ii) the output level?

2. Consider the following macroeconomic model where *m*, *p* and *y* are the logarithms of money supply, the price level and output respectively. (14.4) is a money demand function, (14.5) is a variant of the Phillips curve and (14.6) is the authorities money supply rule. *E* is the expectations operator and ε is a 'white noise' error term.

$$m_t = p_t + y_t \qquad 14.4$$

$$p_t = E_{t-1}p_t + 3(y_t - 50) \qquad 14.5$$

$$m_t = 55 - 0.4(y_{t-1} - 50) + \varepsilon_t \qquad 14.6$$

(a) Obtain a solution for y_t.

(b) Obtain a solution for (i) p_t and (ii) its variance around expected price level.

(c) Can the authorities stabilise output in this model?

(d) Solve for p_t using Lucas method of undetermined coefficients.

Solution

1(a) Note that the *basic method* will suffice for all Rational Expectations models in which there are expectations (at any date in the past) of current events only. The method involves three steps:

1. Solve the model, treating expectations as exogenous.

2. Take the expected value of this solution at the date of the expectations, and solve for the expectations.

3. Substitute the expectations solutions into the solution in 1, and obtain the complete solution.

Substitute equations (14.1) and (14.3) in (14.2) to get the reduced form:

$$\Rightarrow 20 + 0.25(p_t - E_{t-1}p_t) = 30 + 0.25(5 + \varepsilon_t - p_t)$$

$$0.25\varepsilon_t + 11.25 = 0.5p_t - 0.25E_{t-1}p_t \qquad 14.7$$

In order to get $E_{t-1}p_t$ run the expectations operator along equations 14.1-14.3:

$$E_{t-1}y_t = 20 + 0.25(E_{t-1}p_t - E_{t-1}p_t) \equiv 20$$
$$E_{t-1}y_t = 30 + 0.25(E_{t-1}m_t - E_{t-1}p_t)$$
$$E_{t-1}m_t = 5$$

Substituting $E_{t-1}m_t = 5$ and $E_{t-1}y_t = 20$ in the second expression yields solution for $E_{t-1}p_t$.

$$E_{t-1}p_t = 45 \qquad 14.8$$

The full-employment equilibrium value of y^* can be found out by looking at equation (14.1). Note that in equilibrium $p_t - E_{t-1}p_t = 0$. It follows that $y^* = 20$.

Substituting (14.8) in (14.7) yields:

$$\Rightarrow 0.25\varepsilon_t + 11.25 = 0.5p_t - 0.25(45)$$

$$p_t = 45 + 0.5\varepsilon_t \qquad 14.9$$

1(b(i)) If $\varepsilon_t = -2$, substituting in (14.9) gives the price level:

$$p_t = 45 + 0.5(-2) \equiv 44 \qquad 14.10$$

1(b(ii)) Substituting (14.10) and (14.8) in (14.1) yields solution for output:

$$y_t = 20 + 0.25(44 - 45) \equiv 19.75$$

2(a) Substitute equations (14.5) and (14.6) in (14.4) to get the reduced form:

$$55 - 0.4(y_{t-1} - 50) + \varepsilon_t = E_{t-1}p_t + 3(y_t - 50) + y_t \qquad 14.11$$

In order to get $E_{t-1}p_t$ run the expectations operator along equations 14.4-14.6:

$$E_{t-1}m_t = E_{t-1}p_t + E_{t-1}y_t$$
$$E_{t-1}\, p_t = E_{t-1}p_t + 3(E_{t-1}y_t - 50)$$
$$E_{t-1}m_t = 55 - 0.4(y_{t-1} - 50) + E_{t-1}\varepsilon_t \quad \text{(Note } E_{t-1}\varepsilon_t = 0\text{)}$$

Substituting $E_{t-1}m_t = 55 - 0.4(y_{t-1} - 50)$ and $E_{t-1}y_t = 50$ in the first expression yields solution for $E_{t-1}p_t$.

$$E_{t-1}p_t = 25 - 0.4y_{t-1} \qquad 14.12$$

Substituting (14.12) in (14.11) yields solution for output:

$$\Rightarrow 55 - 0.4(y_{t-1} - 50) + \varepsilon_t = 25 - 0.4y_{t-1} + 3(y_t - 50) + y_t$$

$$y_t = 50 + \frac{1}{4}(\varepsilon_t) \qquad 14.13$$

2 (b(i)) Substituting (14.12) and (14.13) in (14.5) yields solution for the price level:

$$\Rightarrow p_t = 25 - 0.4y_{t-1} + 3\left(50 + \frac{1}{4}(\varepsilon_t) - 50\right)$$

Substituting for $y_{t-1}(= 50 + \frac{1}{4}(\varepsilon_{t-1}))$ results in:

$$p_t = 5 + 0.75\varepsilon_t - 0.1\varepsilon_{t-1}$$

2(b(ii)) Since we have the solution for p_t and $E_{t-1}p_t$ it is easy to compute the variance of the price level.

Definition *The variance of X will be denoted by* σ^2 *whether X is a discrete or a continuous type of random variable. It is useful to note that* $\sigma^2 = E[X - \mu]^2 = E[X - 2\mu X + \mu^2]$; *and since E is a linear operator,* $\sigma^2 = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$.

$$E[p_t - E_{t-1}p_t]^2 = E\left[\frac{3}{4}\varepsilon_t\right]^2 = \frac{9}{16}\sigma^2$$

2(c) The authorities cannot stabilise output in this model as output is invariant to the money supply rule. This can be proved by multiplying the money supply equation by (0) instead of (-0.4). The variance of output turns out to be the same ($\frac{1}{16}\sigma^2$) irrespective of the rule in place. This is because the money supply rule is incorporated into agents' expectations at (t-1) and cannot cause any surprises.

2(d) **Lucas method of undetermined coefficients**

Substitute equations (14.5) and (14.6) in (14.4) to get the reduced form:

$$55 - \frac{0.4}{3}(p_{t-1} - E_{t-2}p_{t-1}) + \varepsilon_t = p_t + 50 + \frac{1}{3}(p_t - E_{t-1}p_t)$$

To apply the Lucas method write the solution for the endogenous variables (instead of being written in terms of the constants and the errors) in terms of the 'state variables,' i.e., current and past values of the exogenous variables (including the error terms of the model equations) and past values of the endogenous variables. Thus write the solution for p_t (on which we focus here) as

$$p_t = \pi_0 + \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + \pi_3 p_{t-1} + \pi_4 y_{t-1}$$

By implicaton

$$\begin{aligned} E_{t-1}p_t &= \pi_0 + \pi_2\varepsilon_{t-1} + \pi_3 p_{t-1} + \pi_4 y_{t-1} \\ p_{t-1} &= \pi_0 + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \pi_3 p_{t-2} + \pi_4 y_{t-2} \\ E_{t-2}p_{t-1} &= \pi_0 + \pi_2\varepsilon_{t-2} + \pi_3 p_{t-2} + \pi_4 y_{t-2} \end{aligned}$$

Substituting these conjectures in the reduced form equation and collecting terms in ε_t, ε_{t-1}......

$$\begin{aligned} (\text{cons tan ts}) &: 55 = \pi_0 + 50 \quad \therefore \pi_0 = 5 \\ (\varepsilon_t) &: 1 = \pi_1 + \frac{1}{3}\pi_1 \quad \therefore \pi_1 = \frac{3}{4} \\ (\varepsilon_{t-1}) &: -\frac{0.4}{3}\pi_1 = \pi_2 \quad \therefore \pi_2 = -0.1 \\ (p_{t-1}) &: 0 = \pi_3 \\ (y_{t-1}) &: 0 = \pi_4 \end{aligned}$$

Substituting the $\pi_i^{'}s$ in the conjecture solution for p_t yields:

$$p_t = 5 + 0.75\varepsilon_t - 0.1\varepsilon_{t-1}$$

Tutorial 15

Consider the following model. Fiscal policy will be held constant and monetary policy will be the only policy variable affecting demand for output. For expositional purposes the income velocity of money is also held constant. With these assumptions, the aggregate demand for output can be written in logs as:

$$m_t + \bar{v} = p_t + y_t \qquad 15.1$$

The above equation is the equation of exchange in logs. The model is complete with the introduction of the aggregate supply equation and a money supply rule.

$$y_t = y^* + \alpha(p_t - E_{t-1}p_t) \qquad 15.2$$

$$m_t = \beta y_{t-1} + \varepsilon_t \qquad 15.3$$

(a) Given that agents form expectations rationally, use the basic method to obtain a solution for (i) y_t and (ii) p_t.

(b) If expectations are backward looking i.e., $p_t^e - p_{t-1}^e = \gamma(p_{t-1} - p_{t-1}^e)$, where $0 \prec \gamma \prec 1$, use the basic method to obtain a solution for (i) y_t and (ii) p_t. Is there any scope in this model for the policy authorities to influence output through *systematic* stabilisation policy?

Solution

(a) Substituting equations (15.2) and (15.3) in (15.1) yields the reduced form:

$$\beta y_{t-1} + \varepsilon_t + \bar{v} = p_t + y^* + \alpha(p_t - E_{t-1}p_t) \qquad 15.4$$

In order to solve for $E_{t-1}p_t$ using the basic method:

1. Solve the model, treating expectations as exogenous.

2. Take the expected value of this solution at the date of the expectations, and solve for the expectations.

3. Substitute the expectations solutions into the solution in 1, and obtain the complete solution.

$$E_{t-1}m_t + \bar{v} = E_{t-1}p_t + E_{t-1}y_t \qquad 15.5$$

$$E_{t-1}y_t = y^* + \alpha(E_{t-1}p_t - E_{t-1}p_t) \equiv y^* \qquad 15.6$$

$$E_{t-1}m_t = \beta y_{t-1} + E_{t-1}\varepsilon_t \equiv \beta y_{t-1} \qquad 15.7$$

$$\therefore \ E_{t-1}p_t = \beta y_{t-1} + \bar{v} - y^*$$

Substituting the solution for $E_{t-1}p_t$ in (15.4) yields solution for the price level:

$$\Rightarrow \ \beta y_{t-1} + \varepsilon_t + \bar{v} = p_t + y^* + \alpha(p_t - \beta y_{t-1} - \bar{v} + y^*)$$

$$\Rightarrow \ \beta y_{t-1}(1+\alpha) + \varepsilon_t + \bar{v} - y^* + \alpha\ \bar{v} - \alpha y^* = (1+\alpha)p_t$$

$$\therefore \ p_t = \beta y_{t-1} - y^* + \bar{v} + \left(\frac{1}{1+\alpha}\right)\varepsilon_t$$

Substituting the solutions for p_t and $E_{t-1}p_t$ in (15.2) yields:

$$\Rightarrow \ y_t = y^* + \alpha\left(\beta y_{t-1} - y^* + \bar{v} + \left(\frac{1}{1+\alpha}\right)\varepsilon_t - (\beta y_{t-1} + \bar{v} - y^*)\right)$$

$$y_t = y^* + \left(\frac{1}{1+\alpha}\right)\varepsilon_t$$

The solution for y_t consist of an expected part (y^*) and an unexpected part (functions of ε_t). Rational expectations has incorporated anything known at t-1 with implications for y at time t into the expected part, so that the unexpected part is purely unpredictable.

(b) If expectations are backward looking i.e., $p_t^e - p_{t-1}^e = \gamma(p_{t-1} - p_{t-1}^e)$, where $0 \prec \gamma \prec 1$:

$$\Rightarrow p_t^e = \gamma p_{t-1} - \gamma p_{t-1}^e + p_{t-1}^e$$
$$\Rightarrow p_t^e = \gamma p_{t-1} + (1-\gamma)[\gamma p_{t-2} - \gamma p_{t-2}^e + p_{t-2}^e]$$
$$\Rightarrow p_t^e = \gamma p_{t-1} + \gamma(1-\gamma)p_{t-2} - \gamma(1-\gamma)p_{t-2}^e + (1-\gamma)p_{t-2}^e$$
$$\Rightarrow p_t^e = \gamma p_{t-1} + \gamma(1-\gamma)p_{t-2} + (1-\gamma)^2 p_{t-2}^e$$

By continuous forward substitution for p_{t-2}^e, p_{t-3}^e, ...

$$p_t^e = E_{t-1}p_t = \gamma \sum_{i=0}^{\infty}(1-\gamma)^i p_{t-1+i} \qquad 15.8$$

Substituting (15.8) in (15.4) yields:

$$\Rightarrow \beta y_{t-1} + \varepsilon_t + \bar{v} = p_t + y^* + \alpha\left(p_t - \gamma\sum_{i=0}^{\infty}(1-\gamma)^i p_{t-1+i}\right)$$

$$p_t = \left(\frac{1}{1+\alpha}\right)\left[\beta y_{t-1} + \bar{v} - y^* + \alpha\gamma\sum_{i=0}^{\infty}\left((1-\gamma)^i p_{t-1+i}\right) + \varepsilon_t\right]$$

Substituting p_t and $E_{t-1}p_t$ in (15.2) yields solution for output:

$$y_t = y^* + \alpha\left(\left(\frac{1}{1+\alpha}\right)\left[\begin{array}{c}\beta y_{t-1} + \bar{v} - y^* + \alpha\gamma\sum_{i=0}^{\infty}\left((1-\gamma)^i p_{t-1+i}\right)\\ +\varepsilon_t - \gamma\sum_{i=0}^{\infty}(1-\gamma)^i p_{t-1+i}\end{array}\right]\right)$$

After collecting terms in $\gamma\sum_{i=0}^{\infty}(1-\gamma)^i p_{t-1+i}$ and resorting to some algebraic manipulation we get:

$$y_t = y^* + \left(\frac{\alpha}{1+\alpha}\right)\left[\beta y_{t-1} + \bar{v} - y^* + \varepsilon_t - \gamma\sum_{i=0}^{\infty}(1-\gamma)^i p_{t-1+i}\right]$$

Note that when expectations formation is backward looking (adaptive/error learning) the monetary policy or feedback rule (β -the feedback parameter) enters the solution for output i.e., authorities can stabilise output in this model.

Tutorial 16

1. Consider the following model where symbols have their usual meanings:

$$m_t = p_t + y_t \qquad 16.1$$

$$m_t = \bar{m} - \beta(y_{t-1} - y^*) + \varepsilon_t \qquad 16.2$$

$$y_t = y^* + q\left[p_t - \frac{1}{2}(E_{t-1}p_t + E_{t-2}p_t)\right] \qquad 16.3$$

(a) Using the Muth method, obtain a solution for (i) y_t and (ii) p_t.

(b) Given that agents form expectations rationally, is there any scope in this model for the policy authorities to influence output through *systematic* stabilisation policy?

Solution

1(a) Substitute (16.2) and (16.3) in (16.1) yields:

$$\bar{m} - \beta(y_{t-1} - y^*) + \varepsilon_t = p_t + y^* + q\left[p_t - \frac{1}{2}(E_{t-1}p_t + E_{t-2}p_t)\right] \qquad 16.12$$

It follows from (16.3) that:

$$y_{t-1} = y^* + q\left[p_{t-1} - \frac{1}{2}(E_{t-2}p_{t-1} + E_{t-3}p_{t-1})\right] \qquad 16.13$$

Substituting (16.13) in (16.12) yields:

$$\bar{m} - \beta\left(y^* + q\left[p_{t-1} - \frac{1}{2}(E_{t-2}p_{t-1} + E_{t-3}p_{t-1})\right] - y^*\right) + \varepsilon_t = p_t + y^* +$$

$$q\left[p_t - \frac{1}{2}(E_{t-1}p_t + E_{t-2}p_t)\right]$$

or

$$\bar{m} - \beta(0.5q[(p_{t-1} - E_{t-2}p_{t-1}) + 0.5(p_{t-1} - E_{t-3}p_{t-1})]) + \varepsilon_t = p_t + y^* +$$

$$0.5q[(p_t - E_{t-1}p_t) + (p_t - E_{t-2}p_t)] \qquad 16.14$$

The Muth Method of Undetermined Coefficients:

In a *stochastic linear economic system*, such as the model we are currently working with, a variable can always be written as an infinite moving-average process in a random error. This is Wold's decomposition theorem. We thus write a random variable y_t as:

$$y_t = \bar{y} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

or

$$y_t = \bar{y} + \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ...$$

where $\bar{y}$ = the mean of the series, π_i = constant parameters, and ε = a normally distributed error with a mean of 0, constant variance σ_ε^2 and zero covariance. The same is true for the variable p_t in our model. The Wold's decomposition theorem forms the basis for the Muth method of undetermined coefficients. Using the Muth method we can write the solution for

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results give:

$$\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \\
p_{t-1} &= \pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... \\
E_{t-2}p_{t-1} &= \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... \\
E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \\
E_{t-2}p_t &= \pi_2\varepsilon_{t-2} + ... \\
E_{t-3}p_{t-1} &= \pi_2\varepsilon_{t-3} + ...
\end{aligned}$$

Substituting in (16.14) yields:

$$\Rightarrow \quad \bar{m} - 0.5\beta q(\pi_0\varepsilon_{t-1} + \pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2}) + \varepsilon_t =$$

$$\pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... + y^* + 0.5q[\pi_0\varepsilon_t + \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1}]$$

or

$$\bar{m} - \beta q\pi_0\varepsilon_{t-1} - 0.5\beta q\pi_1\varepsilon_{t-2} + \varepsilon_t =$$

$$\pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... + y^* + q\pi_0\varepsilon_t + 0.5q\pi_1\varepsilon_{t-1} \qquad 16.15$$

We need to now evaluate π_0 and π_1 etc i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \ 1 = \pi_0 + q\pi_0 \Rightarrow \frac{1}{(1+q)}$$

$$\varepsilon_{t-1} : -\beta q\pi_0 = \pi_1 + 0.5q\pi_1$$

$$\Rightarrow -\beta q\frac{1}{(1+q)} = \pi_1 + 0.5q\pi_1$$

$$\therefore \ \pi_1 = \frac{-\beta q}{(1+q)(1+0.5q)}$$

$$\varepsilon_{t-2} : -0.5\beta q\pi_1 = \pi_2$$

$$\therefore \ \pi_2 = \frac{-0.5\beta q(-\beta q)}{(1+q)(1+0.5q)}$$

$$\varepsilon_{t-3} : 0 = \pi_3$$

We know from (16.3) that

$$y_t = y^* + q\left[p_t - \frac{1}{2}(E_{t-1}p_t + E_{t-2}p_t)\right] = y^* + 0.5q[\pi_0\varepsilon_t + \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1}]$$

$$\therefore \;\; y_t = y^* + \frac{q}{1+q}\varepsilon_t - \frac{0.5\beta q^2}{(1+q)(1+0.5q)}\varepsilon_{t-1}$$

1(b) Since β the parameter in the money supply rule enters the solution for output, stabilisation (monetary policy in this case) policy is effective. This illustration shows that in the case of private agents not being able to respond to new information by changing their wage-price decisions as quickly as the monetary authorities can change any of their control, then scope once again emerges for systematic stabilisation policy to have real effects.

Tutorial 17

1. Consider the following model of a closed economy where symbols have their usual meanings. y, p and m are in natural logs. E is the expectations operator and ε is a random disturbance term with the usual properties.

$$y_t = -\alpha r_t \qquad 17.1$$

$$y_t = y^* + \gamma(p_t - E_{t-1}p_t) \qquad 17.2$$

$$y_t = y^* + \gamma(p_t - E_{t-1}p_t) \qquad 17.3$$

$$m_t = \bar{m} + \varepsilon_t \qquad 17.4$$

$$R_t = r_t + E_{t-1}p_{t+1} - E_{t-1}p_t \qquad 17.5$$

(a) Derive the solution for output using the Muth method of undetermined coefficients and comment on the response of output/unemployment to interest rate changes?

(b) Derive the solution for the price level using McCallum's "minimal state variable" criterion?

Solution

1(a) Substituting equation (17.5) the Fisher equation in (17.1) the IS curve yields:

$$\Rightarrow \; y_t = -\alpha(R_t - E_{t-1}p_{t+1} + E_{t-1}p_t)$$

$$R_t = \frac{1}{\alpha}\left(-y_t + \alpha E_{t-1}p_{t+1} - \alpha E_{t-1}p_t \right) \qquad 17.6$$

Substituting equations (17.6) and (17.4) in (17.3) yields:

$$\Rightarrow \; \bar{m} + \varepsilon_t = p_t + y_t - \psi\frac{1}{\alpha}\left(-y_t + \alpha E_{t-1}p_{t+1} - \alpha E_{t-1}p_t \right)$$

Substituting for y_t results in the reduced form equation:

$$\bar{m} + \varepsilon_t = p_t + \left(1 + \frac{\psi}{\alpha}\right)[\gamma((p_t - E_{t-1}p_t) + y^*)] - \frac{\psi}{\alpha}[\alpha E_{t-1}p_{t+1} - \alpha E_{t-1}p_t] \qquad 17.7$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in:

$$\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
p_{t+1} &= \pi_0\varepsilon_{t+1} + \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + \ldots. \\
E_{t-1}p_{t+1} &= \pi_2\varepsilon_{t-1} + \pi_3\varepsilon_{t-2} + \ldots. \\
E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots.
\end{aligned}$$

Substituting all this in (17.7) yields:

$$\bar{m} + \varepsilon_t = \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. + \qquad 17.8$$

$$\left(1 + \frac{\psi}{\alpha}\right)\gamma[\pi_0\varepsilon_t + y^*] - \frac{\psi}{\alpha}\left[\alpha\begin{pmatrix}\pi_2\varepsilon_{t-1} + \\ \pi_3\varepsilon_{t-2} + \ldots.\end{pmatrix} - \alpha\begin{pmatrix}\pi_1\varepsilon_{t-1} + \\ \pi_2\varepsilon_{t-2} + \ldots.\end{pmatrix}\right]$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \; 1 = \pi_0 + \left(1 + \frac{\psi}{\alpha}\right)\gamma\pi_0$$

$$\therefore \; \pi_0 = \frac{1}{1 + \gamma(1 + \frac{\psi}{\alpha})}$$

Note that we do not need $\varepsilon_{t-1}, \varepsilon_{t-2}$ etc for computing the solution for y_t. Substitute the solution for π_0 in (17.2) to get solution for output.

$$y_t = y^* + \gamma(p_t - E_{t-1}p_t) \equiv y^* + \gamma(\pi_0\varepsilon_t)$$

$$\therefore \;\; y_t = y^* + \frac{\gamma}{1 + \gamma(1 + \frac{\psi}{\alpha})}\varepsilon_t$$

Note that the parameter for real interest rate α enters the solution for output. It implies that a rise in the interest rate shifts the aggregate supply curve upwards increasing output. Intuitively, a rise in *r* increases the attractiveness of working today relative to working tomorrow. Thus, there is an increase in employment and output. The response of labour supply to the interest rate is known as the intertemporal substitution in labour supply (Lucas and Rapping, 1969). Equilibrium business cycle theory (which we analyse later) uses the intertemporal substitution of labour to explain why employment and output fluctuates over the business cycle.

1(b) **McCallum's MSV Criterion**

The MSV criterion (see McCallum (1989(a))) is designed to yield a single bubble-free solution by construction. Its definition begins by limiting solutions to those that are linear functions - analogous to our model - of a minimal set of "state variables," i.e., predetermined or exogenous determinants of current endogenous variables. Thus the solution values involve variables that only appear in the model's *structural equations* - no "extraneous" state variables are included, unlike the Lucas method discussed earlier.

It follows that the relevant determinants of p_t include only ε_t and a constant, and we conjecture that the solution is of the form:

$$p_t = \phi_0 + \phi_1\varepsilon_t$$

To find ϕ_0 and $\phi_1,$ we note that if our conjecture solution for p_t is true, then

$$E_{t-1}p_t = \phi_0$$

$$E_{t-1}p_{t+1} = \phi_0$$

Substituting these conjectures in the reduced-form equation (17.7) yields:

$$\bar{m} + \varepsilon_t = \phi_0 + \phi_1\varepsilon_t + \left(1 + \frac{\psi}{\alpha}\right)[\gamma((\phi_1\varepsilon_t) + y^*)] - \frac{\psi}{\alpha}[\alpha\phi_0 - \alpha\phi_0]$$

The following conditions must pertain to the $\phi' s$:

$$\begin{aligned}(\text{constants}):\quad \bar{m} &= \phi_0 + \gamma\left(1 + \frac{\psi}{\alpha}\right)y^* \\ (\varepsilon_t):\quad 1 &= \phi_1 + \gamma\left(1 + \frac{\psi}{\alpha}\right)\phi_1\end{aligned}$$

Now these two conditions - obtained by equating coefficients on both sides of the reduced form are just what is needed, for they permit us to solve for the two unknowns ("undetermined coefficients") - the $\phi' s$. Thus we have,

$$\begin{aligned}\therefore\ \phi_0 &= \bar{m} - \gamma\left(1 + \frac{\psi}{\alpha}\right)y^* \\ \therefore\ \phi_1 &= \frac{1}{1 + \gamma(1 + \frac{\psi}{\alpha})}\end{aligned}$$

Therefore, the solution for p_t is (substituting $\phi' s$ in the conjecture solution for the price level):

$$p_t = \bar{m} - \gamma\left(1 + \frac{\psi}{\alpha}\right)y^* + \frac{1}{1 + \gamma(1 + \frac{\psi}{\alpha})}\varepsilon_t$$

Tutorial 18

1. Consider the following model of a closed economy where symbols have their usual meanings. E is the expectations operator and ε is a random disturbance term with the usual properties.

$$y_t = -\alpha(R_t - E_{t-1}p_{t+1} + E_{t-1}p_t) + \beta G_t - \psi T_t \qquad 18.1$$

$$y_t = y^* + \delta(p_t - E_{t-1}p_t) \qquad 18.2$$

$$G_t = \bar{G} + \varepsilon_t \qquad 18.3$$

$$T_t = \lambda y_t \qquad 18.4$$

where β and ψ are government expenditure and tax elasticities respectively.

(a) Compute the solution for y_t using the Muth method and show that fiscal policy

affects output? If so why?

(b) Compute the variance of output and explain the role of automatic stabilisers? What happens if we have a high tax elasticity?

Solution

(a) Substituting equations 18.2-18.4 in (18.1) yields the reduced form:

$$\Rightarrow y^* + \delta(p_t - E_{t-1}p_t) = -\alpha(R_t - E_{t-1}p_{t+1} + E_{t-1}p_t) + \beta\left(\bar{G} + \varepsilon_t\right) - \psi\lambda y_t$$

or

$$y^* + \delta(p_t - E_{t-1}p_t) = -\alpha(R_t - E_{t-1}p_{t+1} + E_{t-1}p_t) + \beta\left(\bar{G} + \varepsilon_t\right) - \psi\lambda(y^* + \delta(p_t - E_{t-1}p_t))$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in :

$$\begin{aligned} p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \\ p_{t+1} &= \pi_0\varepsilon_{t+1} + \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + ... \\ E_{t-1}p_{t+1} &= \pi_2\varepsilon_{t-1} + \pi_3\varepsilon_{t-2} + ... \\ E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \end{aligned}$$

Plugging in these values in the reduced form equation yields:

$$y^* + \delta(\pi_0\varepsilon_t) = -\alpha\left(\begin{array}{c} R_t - (\pi_2\varepsilon_{t-1} + \pi_3\varepsilon_{t-2} + ...) + \\ (\pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ...) \end{array} \right) +$$

$$\beta\left(\bar{G} + \varepsilon_t\right) - \psi\lambda(y^* + \delta(\pi_0\varepsilon_t))$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \ \delta\pi_0 = \beta - \psi\lambda\delta\pi_0$$

$$\therefore \ \pi_0 = \left(\frac{\beta}{\delta + \psi\lambda\delta} \right)$$

$$\begin{aligned} \varepsilon_{t-1} &: \ 0 = \alpha\pi_2 - \alpha\pi_1 \\ \varepsilon_{t-2} &: \ 0 = \alpha\pi_3 - \alpha\pi_2 \end{aligned}$$

$$\therefore \ \pi_1 = \pi_2 = \pi_3$$

Substituting the value for π_0 in the Phillips curve equation (18.2) yields solution for output:

$$\Rightarrow\ y_t = y^* + \delta(p_t - E_{t-1}p_t) \equiv y^* + \delta(\pi_0\varepsilon_t)$$

$$\therefore\ y_t = y^* + \left(\frac{\delta\beta}{\delta + \psi\lambda\delta}\right)\varepsilon_t \equiv y^* + \left(\frac{\beta}{1 + \psi\lambda}\right)\varepsilon_t$$

$$E_{t-1}y_t = y^* \qquad (\text{since } E_{t-1}\varepsilon_t = 0)$$

Note that both the fiscal instruments (β and ψ are government expenditure and tax elasticities respectively) enter the solution for output. In order to explain why fiscal policy affects output in this model see solution for $E_{t-1}p_t$.

$$E_{t-1}p_t = \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots.$$

Note that π_0 (which contains β and ψ) does not enter the solution for the expected price level. In other words the fiscal instruments are not taken into account when agents' form expectations at t-1 and hence can cause surprises.

b) The variance of output is given by:

$$\Rightarrow \sigma_y^2 = E(y_t - E_{t-1}y_t)^2 = E\left(y^* + \left(\frac{\beta}{1 + \psi\lambda}\right)\varepsilon_t - y^*\right)^2$$

$$\therefore\ \sigma_y^2 = \left(\frac{\beta}{1 + \psi\lambda}\right)^2 \sigma^2$$

Automatic Stabiliser:

By an automatic stabiliser we mean a mechanism in which a variable (say, tax liabilities) responds to current income levels, and therefore provide an automatic and immediate adjustment to current disturbances. Automatic stabilisers are designed to reduce the lags associated with stabilisation policy. The long and variable lags associated with monetary and fiscal policy certainly make macroeconomic management more difficult. Automatic stabilisers are policies that help stimulate or depress the economy when necessary without any deliberate shift in policy stance. In this particular model ψ is the elasticity of spending to temporary variations in tax liabilities. Note that the solution for output is not independent of the automatic stabiliser. A high tax elasticity reduces the variance of output.

Tutorial 19

1. Consider the following model of a closed economy where symbols have their usual meanings. Note that there is persistent shock to output in this model. E is the expectations operator and ε is a random disturbance term with the usual properties.

$$y_t = -\alpha(R_t - E_{t-1}p_{t+1} + E_{t-1}p_t) + \mu_f(y_{t-1} - y^*) \qquad 19.1$$

$$y_t = y^* + \beta(p_t - E_{t-1}p_t) + \gamma(y_{t-1} - y^*) \qquad 19.2$$

$$m_t = p_t + y_t - cR_t + v_t \qquad 19.3$$

$$m_t = \bar{m} + \mu_m(y_{t-1} - y^*) + u_t \qquad 19.4$$

where $\mu_f(y_{t-1} - y^*)$ and $\mu_m(y_{t-1} - y^*)$ represent fiscal and monetary feedback response respectively.

1) Using the Muth method, obtain a solution for y_t.

2) Is there a scope in this model for policy authorities to influence output?

2. Solve the following model for p_t and y_t.

$$m_t = p_t + y_t - \alpha[E_{t-1}p_{t+1} - E_{t-1}p_t] \qquad 19.11$$

$$y_t = y^* + \beta(p_t - E_{t-1}p_t) + \lambda\left(y_{t-1} - y^*\right) \qquad 19.12$$

$$m_t = \bar{m} + \varepsilon_t \qquad 19.13$$

Solve by (a) Sargent's forward root method (b) difference equation method and (c) Muth method.

Solution

1) Note that we can transform (19.2) using the lag operator:

$$\Rightarrow y_t = y^* + \beta(p_t - E_{t-1}p_t) + \gamma(Ly_t - y^*)$$

$$\Rightarrow y_t = y^* + \beta(p_t - E_{t-1}p_t) + \gamma L(y_t - y^*)$$

$$\Rightarrow (1 - \gamma L)(y_t - y^*) = \beta(p_t - E_{t-1}p_t)$$

$$y_t = y^* + \frac{\beta(p_t - E_{t-1}p_t)}{(1-\gamma L)} \qquad 19.5$$

From equation (19.1) one can get the solution for the nominal interest rate:

$$R_t = \frac{1}{\alpha}\left(-y_t + \alpha E_{t-1}p_{t+1} - \alpha E_{t-1}p_t + \mu_f(y_{t-1} - y^*)\right) \qquad 19.6$$

Substituting (19.6) in (19.3) and equating the resulting equation with (19.4) gives:

$$\Rightarrow \quad \bar{m} + \mu_m(y_{t-1} - y^*) + u_t = p_t + y_t - c\frac{1}{\alpha}\begin{pmatrix} -y_t + \alpha E_{t-1}p_{t+1} - \\ \alpha E_{t-1}p_t + \mu_f(y_{t-1} - y^*) \end{pmatrix} + v_t$$

$$\Rightarrow \bar{m} + \mu(y_{t-1} - y^*) + u_t = p_t + \left(1 + \frac{c}{\alpha}\right)y_t - c(E_{t-1}p_{t+1} - E_{t-1}p_t) + v_t$$

where $\mu = [\mu_f \frac{c}{\alpha} + \mu_m]$.

Substituting the solution for output in (19.5) into the reduced form equation yields:

$$\Rightarrow \bar{m} + \mu\left(y^* + \frac{\beta(p_{t-1} - E_{t-2}p_{t-1})}{(1-\gamma L)} - y^*\right) + u_t = p_t +$$

$$\left(1 + \frac{c}{\alpha}\right)\left(y^* + \frac{\beta(p_t - E_{t-1}p_t)}{(1-\gamma L)}\right) - c(E_{t-1}p_{t+1} - E_{t-1}p_t) + v_t$$

or

$$\bar{m}(1-\gamma) + \mu\beta(p_{t-1} - E_{t-2}p_{t-1}) + w_t(1-\gamma L) = p_t(1-\gamma L) + \left(1 + \frac{c}{\alpha}\right)(1-\gamma)y^*$$

$$+ \left(1 + \frac{c}{\alpha}\right)\beta(p_t - E_{t-1}p_t) - c(1-\gamma L)(E_{t-1}p_{t+1} - E_{t-1}p_t)$$

where $w_t = u_t - v_t$.

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in w_t, w_{t-1}, w_{t-2}......

$$w_t \,:\, 1 = \pi_0 + \beta\left(1 + \frac{c}{\alpha}\right)\pi_0$$

$$\therefore\ \pi_0 = \frac{1}{1 + \beta(1 + \frac{c}{\alpha})}$$

The identities in the other errors are irrelevant for our purpose here as it is clear by looking at the solution for output. Substituting the solution for π_0 in (19.5) yields solution for output.

2) After substituting for π_0 in (19.5) we see that π_0 does not depend either on μ_m or μ_f. Thus it is clear that systematic monetary policy does not influence the variance of output in this model inspite of persistence in the aggregate supply curve.

2(a) **Sargent's forward operator method**

Take expectations at t-1 to find $E_{t-1}p_t$ from (19.11-13):

$$\bar{m} = E_{t-1}p_t(1 + \alpha) - \alpha E_{t-1}p_{t+1} + y^* + \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{1 - \lambda L} \qquad 19.14$$

Note that $p_t - E_{t-1}p_t = \pi_0\varepsilon_t$ and $y_{t-1} - y^* = \frac{\beta\pi_0\varepsilon_{t-1}}{1-\lambda L}$.

Hence,

$$\bar{m} = (1 + \alpha)\left(1 - \frac{\alpha}{1 + \alpha}B^{-1}\right)E_{t-1}p_t + y^* + \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{1 - \lambda L} \qquad 19.15$$

where the operator B is defined by $B^{-1}(Ex_{t+j} \mid \Omega_t) = Ex_{t+j+1} \mid \Omega_t$, where Ω_t is the information set at time ‘t’, in other words B instructs to lag the variable while leaving the date of expectations unchanged.

$$\therefore\ E_{t-1}p_t = \frac{\bar{m} - y^*}{(1+\alpha)\left(1 - \frac{\alpha}{1+\alpha}\right)} - \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{(1-\lambda L)(1+\alpha)\left(1 - \frac{\alpha}{1+\alpha}B^{-1}\right)} \qquad 19.16$$

Note the last term (using the summation of an infinite series) can be expressed as:

$$\frac{\lambda\beta\pi_0\varepsilon_{t-1}}{(1-\lambda L)(1+\alpha)\left(1 - \frac{\alpha}{1+\alpha}B^{-1}\right)} = \frac{\lambda\beta\pi_0}{1+\alpha(1-\lambda)}\frac{\varepsilon_{t-1}}{1-\lambda L}$$

Substituting this in (19.16) yields:

$$E_{t-1}p_t = \bar{m} - y^* - \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{(1-\lambda L)(1+\alpha(1-\lambda))} \qquad 19.17$$

It follows that:

$$E_{t-1}p_{t+1} = \bar{m} - y^* - E_{t-1}\frac{\lambda\beta\pi_0\varepsilon_t}{(1-\lambda L)(1+\alpha(1-\lambda))}$$

$$E_{t-1}p_{t+1} = \bar{m} - y^* - \frac{\lambda\beta\pi_0}{1+\alpha(1-\lambda)}E_{t-1}(\varepsilon_t + \lambda\varepsilon_{t-1} + \lambda^2\varepsilon_{t-2} + \ldots\ldots)$$

$$E_{t-1}p_{t+1} = \bar{m} - y^* - \frac{\lambda^2\beta\pi_0}{1+\alpha(1-\lambda)}\left(\frac{\varepsilon_{t-1}}{1-\lambda L}\right) \qquad 19.18$$

Note that the reduced form equation of this model can be expressed as:

$$\bar{m} + \varepsilon_t = (p_t - E_{t-1}p_t) + E_{t-1}p_t - \alpha(E_{t-1}p_{t+1} - E_{t-1}p_t) + y^* + \beta(p_t - E_{t-1}p_t) + \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{1-\lambda L} \qquad 19.19$$

Substituting (19.17) for $E_{t-1}p_t$ and (19.18) for $E_{t-1}p_{t+1}$ in (19.19) and multiplying throughout by $1 - \lambda L$ yields:

$$\varepsilon_t(1 - \pi_0 - \beta\pi_0) - \lambda\varepsilon_{t-1}(1 - \pi_0 - \beta\pi_0) = 0$$

Using the method of undetermined coefficients yields:

$$(\varepsilon_t)\ :\ 1 - \pi_0 - \beta\pi_0 = 0 \qquad \therefore\ \pi_0 = \frac{1}{1+\beta}$$

(ε_{t-1}) : Terms in ε_{t-1} also yield this.

Thus the solutions for p_t and y_t can be expressed as:

$$p_t = \frac{1}{1+\beta}\varepsilon_t + \bar{m} - y^* - \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{(1-\lambda L)(1+\alpha(1-\lambda))}$$

$$p_t = \bar{m} - y^* + \frac{1}{1+\beta}\varepsilon_t - \frac{\lambda}{(1+\alpha(1-\lambda))}(y_{t-1} - y^*)$$

$$y_t = y^* + \frac{\beta}{1+\beta}\varepsilon_t + \lambda(y_{t-1} - y^*)$$

2(b) **Difference equation method**

Take expectations at t-1 to find $E_{t-1}p_t$ from (19.11-13). Rearrange by multiplying throughout by $1 - \lambda L$ to get:

$$(1-\lambda)(\bar{m} - y^*) = -\alpha E_{t-1}p_{t+1} + \alpha\lambda E_{t-2}p_t + (1+\alpha)E_{t-1}p_t - \lambda(1+\alpha)E_{t-2}p_{t-1} + \lambda\pi_0\varepsilon_{t-1}$$

Take expectations at t-2:

$$(1-\lambda)(\bar{m} - y^*) = -\alpha E_{t-2}p_{t+1} + (\alpha\lambda + 1 + \alpha)E_{t-2}p_t - \lambda(1+\alpha)E_{t-2}p_{t-1}$$

Therefore

$$\frac{-(1-\lambda)}{\alpha}(\bar{m} - y^*) = E_{t-2}p_{t+1} - \left(\lambda + \frac{1+\alpha}{\alpha}\right)E_{t-2}p_t + \lambda\left(\frac{1+\alpha}{\alpha}\right)E_{t-2}p_{t-1}$$

or

$$\frac{-(1-\lambda)}{\alpha}(\bar{m} - y^*) = p^e_{t+1} - \left(\lambda + \frac{1+\alpha}{\alpha}\right)p^e_t + \lambda\left(\frac{1+\alpha}{\alpha}\right)p^e_{t-1}$$

where p^e_{t+1} is p_{t+1} expected at t-2. Note that this is a second-order difference equation. Thus the characteristic equation can be expressed as:

$$p^e_{t+i} = A_1\lambda^{i+1} + A_2\left(\frac{1+\alpha}{\alpha}\right)^{i+1} + \bar{m} - y^* \quad (i \geq 1)$$

$$\text{where} \quad A_1 + A_2 = p^e_{t-1} - \bar{m} - y^*$$

$$A_1\lambda + A_2\left(\frac{1+\alpha}{\alpha}\right) = p^e_t - \bar{m} - y^*$$

If we appeal to stability requirement so that $A_2 = 0$, we have

$$A_1 = p^e_{t-1} - (\bar{m} - y^*) \quad \text{and}$$

$$p^e_{t+i} = \bar{m} - y^* + (p^e_{t-1} - (\bar{m} - y^*))\lambda^{i+1} \quad (i \geq 0)$$

Now note that

$$E_{t-1}p_{t+1} = \bar{m} - y^* + \lambda(E_{t-1}p_t - (\bar{m} - y^*))$$

Since

$$E_{t-2}p_t = \bar{m} - y^* + \lambda(E_{t-2}p_{t-1} - (\bar{m} - y^*))$$

Substituting for $E_{t-1}p_{t+1}$ into (19.14) yields:

$$\bar{m} = E_{t-1}p_t(1+\alpha) - \alpha\{\bar{m} - y^* + \lambda(E_{t-1}p_t - (\bar{m} - y^*))\} + y^* + \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{1-\lambda L}$$

or

$$E_{t-1}p_t = \bar{m} - y^* - \frac{\lambda\beta\pi_0\varepsilon_{t-1}}{(1-\lambda L)(1+\alpha(1-\lambda))}$$

Rest of the solution follows from what we had done with Sargent's forward operator method, especially from (19.17) onwards.

2(c) **Muth method**

This is in practice the easiest for this model.

Let $p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$. Substitute directly in (19.19) after multiplying (19.19) by $1-\lambda L$ to get:

$$\begin{aligned}
&(1-\lambda)\bar{m} + \varepsilon_t - \lambda\varepsilon_{t-1} \\
&= \bar{p} + \sum_{i=0}^{\infty}\pi_i\varepsilon_{t-i} - \lambda\left(\bar{p} + \sum_{i=1}^{\infty}\pi_{i-1}\varepsilon_{t-i}\right) - \alpha\left(\sum_{i=1}^{\infty}\pi_{i+1}\varepsilon_{t-i} - \sum_{i=1}^{\infty}\pi_i\varepsilon_{t-i}\right) \\
&\quad + \alpha\lambda\left(\sum_{i=2}^{\infty}\pi_i\varepsilon_{t-i} - \sum_{i=2}^{\infty}\pi_{i-1}\varepsilon_{t-i}\right) + (1-\lambda)y^* + \beta\pi_0\varepsilon_t
\end{aligned}$$

Using the method of undetermined coefficients yields:

Collecting terms in:

$$\begin{aligned}
(\text{constants}) &: (1-\lambda)\bar{m} = (1-\lambda)\bar{p} + (1-\lambda)y^* \quad \therefore \quad \bar{p} = \bar{m} - y^* \\
(\varepsilon_t) &: 1 = \pi_0 + \beta\pi_0 \qquad \therefore \quad \pi_0 = \frac{1}{1+\beta} \\
(\varepsilon_{t-1}) &: -\lambda = \pi_1 - \lambda\pi_0 - \alpha(\pi_2 - \pi_1) \\
(\varepsilon_{t-i},\ i \geq 2) &: 0 = -\alpha\pi_{i+1} + (1+\alpha+\alpha\lambda)\pi_i - \lambda(1+\alpha)\pi_{i-1}
\end{aligned}$$

or

$$0 = \pi_{i+1} - \left(\frac{1+\alpha}{\alpha} + \lambda\right)\pi_i + \lambda\left(\frac{1+\alpha}{\alpha}\right)\pi_{i-1}$$

This is a difference equation in π_i, initial values $(i = 2)$. Thus the characteristic equation can be expressed as:

$$\pi_i = A_1\lambda^{i-1} + A_2\left(\frac{1+\alpha}{\alpha}\right)^{i-1} + \bar{m} - y^* \qquad (i \geq 3) \qquad \text{where}$$
$$\pi_1 = A_1 + A_2$$
$$\pi_2 = A_1\lambda + A_2\left(\frac{1+\alpha}{\alpha}\right)$$

Appeal to stability requirement to set $A_2 = 0$, hence

$$\pi_1 = A_1;\, \pi_2 = \lambda\pi_1$$

Substitute $\pi_2 = \lambda\pi_1$ in $(\varepsilon_{t-1}) : -\lambda = \pi_1 - \lambda\pi_0 - \alpha(\lambda\pi_1 - \pi_1)$ to get

$$\pi_1 = \frac{\lambda\left(\frac{-\beta}{1+\beta}\right)}{(1+\alpha(1-\lambda))}$$

Thus the solutions for p_t and y_t is:

$$p_t = \bar{m} - y^* + \frac{1}{1+\beta}\varepsilon_t - \frac{\lambda\beta}{(1+\beta)(1+\alpha(1-\lambda))}(\varepsilon_{t-1} + \lambda\varepsilon_{t-2} + \ldots\ldots)$$
$$p_t = \bar{m} - y^* + \frac{1}{1+\beta}\varepsilon_t - \frac{\lambda}{(1+\alpha(1-\lambda))}(y_{t-1} - y^*)$$
$$y_t = y^* + \frac{\beta}{1+\beta}\varepsilon_t + \lambda(y_{t-1} - y^*)$$

Tutorial 20

Consider the following model in natural logarithms.

$$\bar{m} - p_t = \alpha_0 + \alpha_1 y_t + \alpha_2 R_t \qquad 20.1$$

$$y_t = y^* + \delta(p_t - E_{t-1} p_t) \qquad 20.2$$

$$R_t = (p_{t-1} - p_{t-2}) + \gamma(y_{t-1} - y^*) + \beta[(p_{t-1} - p_{t-2}) - 2.5] + 2.5 + \varepsilon_t \qquad 20.3$$

where $\gamma = \beta = 0.5$ is the weight placed on the deviation of output and inflation rate from target. Equation (20.3) is a simplified version of a backward looking Taylor rule (see Taylor, 1993) which captures the spirit of recent research in macroeconomics.

R_t = Nominal interest rate

$p_{t-1} - p_{t-2}$ = inflation rate in t-1

y^* = potential real GDP

$\bar{m}$ = constant money supply growth

ε_t = white noise error

The policy rule in equation (20.3) has the feature that the nominal rate rises if inflation increases above a target of 2.5% (set by the Chancellor for the MPC in 1997) or if real GDP rises above potential GDP. If both the inflation rate and the real GDP are on target, then the nominal rate would equal 5% or 2.5% in real terms.

1. Solve the model using the Muth method. Can authorities stabilise output in this model if they use a Taylor rule instead of an explicit money supply rule?

2. Would our conclusions change if we had a New Keynesian Phillips curve of the form $y_t = y^* + 0.5q[(p_t - E_{t-1} p_t) + (p_t - E_{t-2} p_t)]$ instead?

Solution

1) Substituting equations (20.3) and (20.2) in (20.1) yields the reduced form equation:

$$\Rightarrow \bar{m} - p_t = \alpha_0 + \alpha_1(y^* + \delta(p_t - E_{t-1}p_t)) +$$

$$\alpha_2\left[(p_{t-1} - p_{t-2}) + \gamma(y_{t-1} - y^*) + \beta[(p_{t-1} - p_{t-2}) - 2.5] + 2.5 + \varepsilon_t\right]$$

or

$$\bar{m} - p_t = \alpha_0 + \alpha_1(y^* + \delta(p_t - E_{t-1}p_t)) +$$

$$\alpha_2\left[\begin{array}{c}(p_{t-1} - p_{t-2}) + \gamma(y^* + \delta(p_{t-1} - E_{t-2}p_{t-1}) - y^*) + \\ \beta[(p_{t-1} - p_{t-2}) - 2.5] + 2.5 + \varepsilon_t\end{array}\right]$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in :

$$\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
p_{t-1} &= \pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + \ldots. \\
E_{t-2}p_{t-1} &= \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + \ldots. \\
E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
p_{t-2} &= \pi_0\varepsilon_{t-2} + \pi_1\varepsilon_{t-3} + \pi_2\varepsilon_{t-4} + \ldots. \\
E_{t-3}p_{t-1} &= \pi_2\varepsilon_{t-3} + \ldots.
\end{aligned}$$

Plugging in these values in the reduced form equation yields:

$$\bar{m} - (\pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ...) = \alpha_0 + \alpha_1(y^* + \delta(\pi_0\varepsilon_t)) +$$

$$\alpha_2\left[(1+\beta)\left(\pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... - \left(\begin{array}{c} \pi_0\varepsilon_{t-2} + \pi_1\varepsilon_{t-3} + \\ \pi_2\varepsilon_{t-4} + ... \end{array} \right)\right)\right] +$$

$$\alpha_2\gamma\delta\pi_0\varepsilon_{t-1} - \alpha_2\beta(2.5) + \alpha_2(2.5) + \alpha_2\varepsilon_t$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \quad -\pi_0 = \alpha_1\delta\pi_0 + \alpha_2$$

$$\therefore \ \pi_0 = -\left(\frac{\alpha_2}{1+\alpha_1\delta} \right)$$

$$\varepsilon_{t-1} : -\pi_1 = \alpha_2(1+\beta)\pi_0 + \alpha_2\gamma\delta\pi_0$$

$$\therefore \ \pi_1 = \frac{(\alpha_2)^2}{1+\alpha_1\delta}(1+\beta+\gamma\delta)$$

The identities in the other errors are irrelevant for computing the solution for output. Substituting the solution for π_0 in (20.2) yields solution for output:

$$y_t = y^* + \delta(\pi_0\varepsilon_t) \equiv y^* - \delta\left(\frac{\alpha_2}{1+\alpha_1\delta} \right)\varepsilon_t$$

Note that the parameters of the interest rate rule do not enter the solution for output. Therefore authorities cannot stabilise output in this model. However, if one substitutes for π_0, π_1,... in the price level equation $\left(p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i\varepsilon_{t-i}\right)$ it is clear that β and γ (parameters in the interest rate rule) enter π_1 (see above). It is clear that the interest rate rule does affect the solution for the price level.

2) Suppose we had a New Keynesian Phillips curve, our reduced form equation becomes:

$$\bar{m} - p_t = \alpha_0 + \alpha_1(y^* + 0.5q[(p_t - E_{t-1}p_t) + (p_t - E_{t-2}p_t)]) +$$

$$\alpha_2 \begin{bmatrix} (p_{t-1} - p_{t-2}) + \gamma(0.5q[(p_{t-1} - E_{t-2}p_{t-1}) + (p_{t-1} - E_{t-3}p_{t-1})]) + \\ \beta[(p_{t-1} - p_{t-2}) - 2.5] + 2.5 + \varepsilon_t \end{bmatrix}$$

Plugging in these values in the reduced form equation yields:

$$\bar{m} - (\pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ...) = \alpha_0 + \alpha_1\begin{pmatrix} y^* + q\pi_0\varepsilon_t + \\ 0.5q\pi_1\varepsilon_{t-1} \end{pmatrix} +$$

$$\alpha_2\left[(1+\beta)\left(\pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... - \begin{pmatrix} \pi_0\varepsilon_{t-2} + \pi_1\varepsilon_{t-3} + \\ \pi_2\varepsilon_{t-4} + ... \end{pmatrix} \right) \right] +$$

$$\alpha_2\gamma q\pi_0\varepsilon_{t-1} + \alpha_2\gamma 0.5q\pi_1\varepsilon_{t-2} - \alpha_2\beta(2.5) + \alpha_2(2.5) + \alpha_2\varepsilon_t$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \quad -\pi_0 = \alpha_1 q\pi_0 + \alpha_2$$

$$\therefore \pi_0 = -\left(\frac{\alpha_2}{1 + \alpha_1 q}\right)$$

$$\varepsilon_{t-1} : -\pi_1 = \alpha_1 0.5q\pi_1 + \alpha_2(1+\beta)\pi_0 + \alpha_2\gamma q\pi_0$$

$$\therefore \pi_1 = \frac{(\alpha_2)^2(1 + \beta + \gamma q)}{(1 + \alpha_1 q)(1 + \alpha_1 0.5q)}$$

Note that we do not need $\varepsilon_{t-2}, \varepsilon_{t-3}$ etc for computing the solution for y_t. Now substitute the solution for π_0 and π_1 in the New Keynesian Phillips curve to get solution for output.

$$y_t = y^* + q\pi_0\varepsilon_t + 0.5q\pi_1\varepsilon_{t-1}$$

Note that the parameters of the Taylor rule β and γ enter the coefficient π_1 and hence the solution for output. Thus, it is clear that short-run non-neutrality depends on what sort of aggregate supply curve is in place and not on the way monetary policy (money supply/interest rate rule) is conducted.

Tutorial 21

Consider a version of Cagan (1956) model of hyperinflation. All symbols are interpreted as being deviations from equilibrium. γ is the policy parameter describing the direction and degree of intervention. For example, a value of $\gamma \succ 0$ means authorities seek to offset the higher inflationary pressures by engaging in monetary contraction.

$$m_t - p_t = -\beta[E_t p_{t+1} - p_t] + \varepsilon_t \qquad 21.1$$

$$m_t = -\gamma p_t \qquad 21.2$$

1. Comment on the solution path for p_t by using the Muth method?

2. What happens when we impose the terminal condition $\pi_1 = 0$ on the solution path for the price level?

Solution

1) Substituting equation (21.2) in (21.1) yields:

$$\Rightarrow -\gamma p_t - p_t = -\beta[E_t p_{t+1} - p_t] + \varepsilon_t$$

$$\Rightarrow -p_t(\gamma + 1 + \beta) - \varepsilon_t = -\beta(E_t p_{t+1})$$

$$\beta(E_t p_{t+1}) = (1 + \beta + \gamma)p_t + \varepsilon_t$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in :

$$
\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
p_{t+1} &= \pi_0\varepsilon_{t+1} + \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + \ldots. \\
E_t p_{t+1} &= \pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + \ldots.
\end{aligned}
$$

Plugging in these values in the reduced form equation yields:

$$
\beta(\pi_1\varepsilon_t + \pi_2\varepsilon_{t-1} + \ldots.) = (1+\beta+\gamma)\begin{pmatrix} \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \\ \pi_2\varepsilon_{t-2} + \ldots. \end{pmatrix} + \varepsilon_t
$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in ε_t, ε_{t-1}, ε_{t-2}......

$$
\varepsilon_t : \ \beta\pi_1 = (1+\beta+\gamma)\pi_0 + 1
$$

$$
\therefore \ \pi_0 = \frac{\beta\pi_1 - 1}{(1+\beta+\gamma)}
$$

$$
\varepsilon_{t-1} : \ \beta\pi_2 = (1+\beta+\gamma)\pi_1
$$

$$
\therefore \ \pi_1 = \frac{\beta\pi_2}{(1+\beta+\gamma)}
$$

or

$$
\pi_2 = \left(\frac{(1+\beta+\gamma)}{\beta}\right)\pi_1
$$

$$
\varepsilon_{t-2} : \ \beta\pi_3 = (1+\beta+\gamma)\pi_2
$$

$$\therefore\ \pi_2 = \frac{\beta\pi_3}{(1+\beta+\gamma)}$$

or

$$\pi_3 = \left(\frac{(1+\beta+\gamma)}{\beta}\right)\pi_2$$

Similarly,

$$\pi_{i+1} = \left(\frac{(1+\beta+\gamma)}{\beta}\right)\pi_i \qquad i = 1, 2.........$$

Therefore, the solution for p_t is:

$$\Rightarrow p_t = \bar{p} + \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ...$$

$$p_t = \bar{p} + \left(\frac{\beta\pi_1 - 1}{(1+\beta+\gamma)}\right)\varepsilon_t + \pi_1\left[\begin{array}{c}\varepsilon_{t-1} + \left(\frac{(1+\beta+\gamma)}{\beta}\right)\varepsilon_{t-2} + \\ \left(\frac{(1+\beta+\gamma)}{\beta}\right)^2\varepsilon_{t-3} + ...\end{array}\right]$$

where π_1 is undetermined. This model clearly has the saddle path property i.e., there is an infinity of paths/multiplicity of arbitrary solutions exist. One way to resolve this issue is to impose a terminal condition $\pi_1 = 0$ on the solution path in order to rule out a bubble.

Tutorial 22

Consider the following model (a modified version of Taylor (1977)) where symbols have their usual meanings.

$$y_t = -\beta_1[R_t - (E_{t-1}p_{t+1} - E_{t-1}p_t)] + \beta_2(m_t - p_t) + u_{1t} \qquad 22.1$$

$$m_t - p_t = y_t - \alpha_1 R_t + \alpha_2(m_t - p_t) + u_{2t} \qquad 22.2$$

$$y_t = \gamma_0 + \gamma_1(m_t - p_t) + u_{3t} \qquad 22.3$$

$$m_t = m \qquad 22.4$$

1. Determine the solution for p_t using the Muth method and comment on the solution?

2. What sort of terminal condition is required to get a stable solution?

Solution

1) Our first objective is to get the reduced form equation. Note that we can get R_t from equation (22.2).

$$\Rightarrow -\alpha_1 R_t = (1 - \alpha_2)(m_t - p_t) - y_t - u_{2t}$$

$$R_t = \frac{1}{\alpha_1}[y_t - (1 - \alpha_2)(m_t - p_t) + u_{2t}] \qquad 22.5$$

Substituting (22.5), (22.3) and (22.4) in (22.1) yields the reduced form equation:

$$\Rightarrow \gamma_0 + \gamma_1(m - p_t) + u_{3t} =$$

$$-\beta_1\left[\begin{array}{c} \frac{1}{\alpha_1}(\gamma_0 + \gamma_1(m - p_t) + u_{3t} - (1 - \alpha_2)(m - p_t) + u_{2t}) - \\ (E_{t-1}p_{t+1} - E_{t-1}p_t) \end{array}\right] +$$

$$\beta_2(m - p_t) + u_{1t}$$

Collecting terms in $(m - p_t)$ yields:

$$\Rightarrow \gamma_0 + \left[\gamma_1 + \frac{\beta_1}{\alpha_1}\gamma_1 - \frac{\beta_1}{\alpha_1}(1-\alpha_2) - \beta_2 \right](m - p_t) + u_{3t} =$$
$$-\frac{\beta_1}{\alpha_1}[\gamma_0 + u_{3t} + u_{2t}] +$$
$$\beta_1 E_{t-1}p_{t+1} - \beta_1 E_{t-1}p_t + u_{1t}$$

or

$$E_{t-1}p_{t+1} = E_{t-1}p_t + \delta_1 p_t + \delta_0 + u_t$$

where

$$\delta_1 = \frac{(1-\alpha_2)}{\alpha_1} + \frac{\beta_2}{\beta_1} - \gamma_1\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right)$$
$$\delta_0 = \gamma_0\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right) - \delta_1 m$$
$$u_t = -\left(\frac{1}{\beta_1}\right)u_{1t} + \left(\frac{1}{\alpha_1}\right)u_{2t} + \left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right)u_{3t}$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i u_{t-i}$$

Ignoring the constants and expanding results in :

$$p_t = \pi_0 u_t + \pi_1 u_{t-1} + \pi_2 u_{t-2} + \ldots.$$
$$E_{t-1}p_t = \pi_1 u_{t-1} + \pi_2 u_{t-2} + \pi_3 u_{t-3} + \ldots.$$
$$p_{t+1} = \pi_0 u_{t+1} + \pi_1 u_t + \pi_2 u_{t-1} + \ldots.$$
$$E_{t-1}p_{t+1} = \pi_2 u_{t-1} + \pi_3 u_{t-2} + \ldots.$$

Plugging in these values in the reduced form equation yields:

$$\Rightarrow \pi_2 u_{t-1} + \pi_3 u_{t-2} + ... = \pi_1 u_{t-1} + \pi_2 u_{t-2} + \pi_3 u_{t-3} + ... +$$

$$\delta_1(\pi_0 u_t + \pi_1 u_{t-1} + \pi_2 u_{t-2} + ...) + \delta_0 + u_t$$

We need to now evaluate π_0 and π_1 i.e., the undetermined coefficients. For this we collect terms in u_t, u_{t-1}, u_{t-2}......

$$u_t : \; 0 = \delta_1 \pi_0 + 1$$

$$\therefore \; \pi_0 = -\frac{1}{\delta_1}$$

$$u_{t-1} : \; \pi_2 = \pi_1 + \delta_1 \pi_1$$

$$\therefore \; \pi_1 = \frac{\pi_2}{(1 + \delta_1)}$$

or

$$\pi_2 = (1 + \delta_1)\pi_1$$

$$u_{t-2} : \; \pi_3 = \pi_2 + \delta_1 \pi_2$$

$$\therefore \; \pi_2 = \frac{\pi_3}{(1 + \delta_1)}$$

or

$$\pi_3 = (1 + \delta_1)\pi_2$$

Similarly,

$$\pi_{i+1} = (1+\delta_1)\pi_i \qquad i = 1,2.........$$

Therefore, the solution for p_t is:

$$\Rightarrow p_t = \bar{p} + \pi_0 u_t + \pi_1 u_{t-1} + \pi_2 u_{t-2} + ...$$

$$p_t = -\left(\frac{\delta_0}{\delta_1}\right) - \left(\frac{1}{\delta_1}\right)u_t + \pi_1 \sum_{i=0}^{\infty}(1+\delta_1)^i u_{t-i-1}$$

where π_1 remains undetermined.

2) This model therefore has the saddlepath property i.e., there is an infinity of paths all but one unstable. This immediately raises the question of which solution to choose. Taylor resolves this by proposing the criteria that variance of the price level be minimised. This implies $\pi_1 = 0$, so that the solution for p_t reduces to:

$$p_t = -\left(\frac{\delta_0 + u_t}{\delta_1}\right)$$

Here, the terminal condition ($\pi_1 = 0$) will select the *most stable* solution by, in effect, ruling out the root with the largest modulus.

Tutorial 23

1. Consider the following model (with future expectations) where symbols have their usual meanings.

$$m_t = p_t + y_t - \alpha[E_{t-1}p_{t+1} - E_{t-1}p_t] \qquad 23.1$$

$$y_t = y^* + \beta(p_t - E_{t-1}p_t) \qquad 23.2$$

$$m_t = \bar{m} + \varepsilon_t \qquad 23.3$$

Determine the solution for p_t and y_t using the method of forward substitution and comment on the solution?

2. Consider the following model where symbols have their usual meanings.

$$m_t = p_t + y_t \qquad 23.11$$

$$y_t = y^* + \beta\{p_t - 0.5(E_{t-1}p_t + E_{t-2}p_t)\} + \lambda(y_{t-1} - y^*) \qquad 23.12$$

$$m_t = \bar{m} + \mu(y_{t-1} - y^*) + \varepsilon_t \qquad 23.13$$

Determine the solution for p_t and y_t using the Muth method of undetermined coefficients?

Solution

1) Substituting (23.3) and (23.2) in (23.1) yields the reduced form equation:

$$\bar{m} + \varepsilon_t = p_t + y^* + \beta(p_t - E_{t-1}p_t) - \alpha[E_{t-1}p_{t+1} - E_{t-1}p_t]$$

In order to solve for $E_{t-1}p_t$ using the basic method:

1. Solve the model, treating expectations as exogenous.

2. Take the expected value of this solution at the date of the expectations, and solve for

the expectations.

3. Substitute the expectations solutions into the solution in 1, and obtain the complete solution.

$$E_{t-1}m_t = E_{t-1}p_t + E_{t-1}y_t - \alpha[E_{t-1}p_{t+1} - E_{t-1}p_t] \qquad 23.4$$

$$E_{t-1}y_t = y^* + \beta(E_{t-1}p_t - E_{t-1}p_t) \equiv y^* \qquad 23.5$$

$$E_{t-1}m_t = \bar{m} + E_{t-1}\varepsilon_t \equiv \bar{m} \qquad 23.6$$

Substituting (23.5) and (23.6) in (23.4) yields:

$$E_{t-1}p_t = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}p_{t+1} \qquad 23.7$$

Note that this is not the solution for $E_{t-1}p_t$ because $E_{t-1}p_{t+1}$ is not solved out; we have shifted the problem into the future. To solve for $E_{t-1}p_{t+1}$ we lead the model by one period:

$$E_t p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_t p_{t+2} \qquad 23.8$$

Leading it further by one period yields:

$$E_{t+1}p_{t+2} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t+1}p_{t+3} \qquad 23.9$$

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Taking expectations of (23.8) and (23.9) at time t-1 yields:

$$E_{t-1}E_t p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}E_t p_{t+2} \qquad 23.10$$

$$E_{t-1}E_{t+1}p_{t+2} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}E_{t+1}p_{t+3} \qquad 23.11$$

At this point we can invoke the Law of Iterated Expectations i.e.,

$$E_{t-1}E_t p_{t+1} = E_{t-1}p_{t+1}$$
$$E_{t-1}E_{t+1}p_{t+2} = E_{t-1}p_{t+2}$$

Hence, we can express (23.10) as follows:

$$E_{t-1}p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}p_{t+2}$$

Substituting successively (forward) for $E_{t-1}p_{t+2}$, $E_{t-1}p_{t+3}$ and so on results in:

$$\Rightarrow E_{t-1}p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)\left[\frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}p_{t+3}\right]$$
$$\Rightarrow E_{t-1}p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)\left(\frac{\bar{m} - y^*}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^2$$
$$\times\left(\frac{\bar{m} - y^*}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^3 E_{t-1}p_{t+4}$$

or

$$E_{t-1}p_{t+1} = \frac{1}{1+\alpha}\sum_{i=0}^{N-1}\left(\frac{\alpha}{1+\alpha}\right)^i(\bar{m} - y^*) + \left(\frac{\alpha}{1+\alpha}\right)^N E_{t-1}p_{t+N+1} \qquad 23.12$$

As $N \to \infty$ and applying the stability condition to $E_{t-1}p_{t+i}$,
so that $E_{t-1}p_{t+N+1} \to 0$ as $N \to \infty$.

Thus we can write (23.12) as:

$$\Rightarrow E_{t-1}p_{t+1} = \frac{1}{1+\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^i(\bar{m} - y^*)$$

$$E_{t-1}p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha}\left[1 + \left(\frac{\alpha}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^2 + \left(\frac{\alpha}{1+\alpha}\right)^3 + ...\right]$$

(Summation of an infinite series)

$$E_{t-1}p_{t+1} = \frac{\bar{m} - y^*}{1+\alpha}\left[\frac{1}{1-\frac{\alpha}{1+\alpha}}\right] \equiv \bar{m} - y^*$$

By the same argument but starting from (23.7) we get:

$$\Rightarrow \; E_{t-1}p_t = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)\left[\frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)E_{t-1}p_{t+2}\right]$$

$$\Rightarrow E_{t-1}p_t = \frac{\bar{m} - y^*}{1+\alpha} + \left(\frac{\alpha}{1+\alpha}\right)\left(\frac{\bar{m} - y^*}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\bar{m} - y^*}{1+\alpha}\right)$$

$$+ \left(\frac{\alpha}{1+\alpha}\right)^3 E_{t-1}p_{t+3}$$

or

$$E_{t-1}p_t = \frac{\bar{m} - y^*}{1+\alpha}\sum_{i=0}^{N-1}\left(\frac{\alpha}{1+\alpha}\right)^i + \left(\frac{\alpha}{1+\alpha}\right)^N E_{t-1}p_{t+N+1}$$

$$\therefore \; E_{t-1}p_t = \bar{m} - y^*$$

Note that the paths for events can be unstable. Our model here implies that all paths for prices, except that for which $E_{t-1}p_t = \bar{m} - y^*$, explode monotonically. Prices would be propelled into either ever-deepening hyperdeflation or ever-accelerating hyperinflation, even though money supply is held rigid. So we need to impose a *terminal condition* for selecting a unique stable path. In this model we use the method of forward substitution in order to ensure convergence. Substituting $E_{t-1}p_t$ and $E_{t-1}p_{t+1}$ in the reduced form equation yields solution for the price level:

$$\Rightarrow \bar{m} + \varepsilon_t \; = p_t + y^* + \beta(p_t - (\bar{m} - y^*)) - \alpha[(\bar{m} - y^*) - (\bar{m} - y^*)]$$

or

$$p_t = \bar{m} - y^* + \left(\frac{1}{1+\beta}\right)\varepsilon_t$$

Substituting the solution for p_t and $E_{t-1}p_t$ in the Phillips curve equation (23.2) yields solution for output.

$$y_t = y^* + \left(\frac{\beta}{1+\beta}\right)\varepsilon_t$$

2) Note that (23.12) can be expressed as

$$y_t - y^* = \frac{\beta(\pi_0\varepsilon_t + 0.5\pi_1\varepsilon_{t-1})}{1-\lambda L}$$

The reduced form equation can be expressed as:

$$\bar{m} + \frac{\mu\beta(\pi_0\varepsilon_{t-1} + 0.5\pi_1\varepsilon_{t-2})}{1-\lambda L} + \varepsilon_t = \sum_{i=0}^{\infty}\pi_i\varepsilon_{t-i} + \bar{p} + y^* + \frac{\beta(\pi_0\varepsilon_t + 0.5\pi_1\varepsilon_{t-1})}{1-\lambda L}$$

Multiplying throughout by $1-\lambda L$ yields:

$$(1-\lambda)(\bar{m} - y^*) + \mu\beta(\pi_0\varepsilon_{t-1} + 0.5\pi_1\varepsilon_{t-2}) + \varepsilon_t - \lambda\varepsilon_{t-1}$$
$$= (1-\lambda)\,\bar{p} + \sum_{i=0}^{\infty}\pi_i\varepsilon_{t-i} - \lambda\sum_{i=1}^{\infty}\pi_{i-1}\varepsilon_{t-i} + \beta(\pi_0\varepsilon_t + 0.5\pi_1\varepsilon_{t-1})$$

Collecting terms in:

$$\begin{aligned}
(\text{constants}) &: (1-\lambda)(\bar{m} - y^*) = (1-\lambda)\,\bar{p} \quad \therefore \quad \bar{p} = \bar{m} - y^* \\
(\varepsilon_t) &: 1 = \pi_0 + \beta\,\pi_0 \qquad \therefore \quad \pi_0 = \frac{1}{1+\beta} \\
(\varepsilon_{t-1}) &: \mu\beta\pi_0 - \lambda = \pi_1 - \lambda\pi_0 + \beta(0.5\pi_1) \\
&\therefore \pi_1 = \frac{\mu\beta + \lambda(1-\{1+\beta\})}{(1+0.5\beta)(1+\beta)} \\
(\varepsilon_{t-2}) &: 0.5\mu\beta\pi_1 = \pi_2 - \lambda\pi_1 \qquad \therefore \quad \pi_2 = (0.5\mu\beta + \lambda)\pi_1 \\
(\varepsilon_{t-i},\ i \geq 3) &: 0 = \pi_i - \lambda\pi_{i-1}
\end{aligned}$$

Therefore the solution for p_t and y_t can be expressed as:

$$p_t = \bar{m} - y^* + \frac{1}{1+\beta}\varepsilon_t + \frac{\mu\beta + \lambda(1-\{1+\beta\})}{(1+0.5\beta)(1+\beta)}\varepsilon_{t-1} + \frac{(0.5\mu\beta + \lambda)\pi_1\varepsilon_{t-2}}{1-\lambda L}$$

$$y_t = y^* + \frac{\frac{\beta}{1+\beta}\varepsilon_t + 0.5\beta\frac{\mu\beta+\lambda(1-\{1+\beta\})}{(1+0.5\beta)(1+\beta)}\varepsilon_{t-1}}{1-\lambda L}$$

Tutorial 24

Lucas Critique of Econometric policy evaluation:

Consider the following model where symbols have their usual meaning.

$$y_t = \beta[p_t - 0.5(E_{t-1}p_t + E_{t-2}p_t)] \qquad 24.1$$

$$m_t = p_t + y_t \qquad 24.2$$

$$m_t = \mu y_{t-1} + \varepsilon_t \qquad 24.3$$

1. What is the solution for y_t in terms of current and past ε_t?

2. What is optimal μ?

3. What is the solution for y_t in terms of current and lagged m_t and y_t?

4. If this was treated as the basis for calculating optimal m_t, what would this be?

5. If this money supply (m_t) rule was implemented, what would then be the solution for output in terms of current and lagged ε_t?

Solution

1) Substituting equations (24.1) and (24.3) in (24.2) yields the reduced form equation:

$$\mu\beta(p_{t-1} - 0.5(E_{t-2}p_{t-1} + E_{t-3}p_{t-1})) + \varepsilon_t = p_t + \beta\begin{bmatrix} p_t - 0.5 \\ (E_{t-1}p_t + E_{t-2}p_t) \end{bmatrix} \qquad 24.4$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in :

$$\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + \ldots. \\
E_{t-2}p_t &= \pi_2\varepsilon_{t-2} + \pi_3\varepsilon_{t-3} + \ldots. \\
p_{t-1} &= \pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + \ldots. \\
E_{t-2}p_{t-1} &= \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + \ldots. \\
E_{t-3}p_{t-1} &= \pi_2\varepsilon_{t-3} + \pi_3\varepsilon_{t-4} + \ldots.
\end{aligned}$$

Plugging in these values in the reduced form equation and collecting terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t \; : \; 1 = \pi_0 + \beta\pi_0$$

$$\therefore \; \pi_0 = \left(\frac{1}{1+\beta} \right)$$

$$\varepsilon_{t-1} \; : \; \mu\beta\pi_0 = \pi_1 + \beta[\pi_1 - 0.5\pi_1]$$

$$\therefore \; \pi_1 = \frac{\mu\beta\pi_0}{1+0.5\beta} \equiv \frac{\mu\beta}{(1+\beta)(1+0.5\beta)}$$

$$\varepsilon_{t-2} \; : \; \mu\beta(\pi_1 - 0.5\pi_1) = \pi_2 + \beta[\pi_2 - 0.5(\pi_2 + \pi_2)]$$

$$\therefore \; \pi_2 = \frac{1}{2}\mu\beta\pi_1$$

$$\pi_3 = \pi_4 = \pi_5 = 0$$

The identities in the other errors are irrelevant for computing the solution for output. Substituting the solution for π_0 and π_1 in the New Keynesian Phillips curve yields solution for output in terms of current and past ε_t:

$$\Rightarrow y_t = \beta[p_t - 0.5(E_{t-1}p_t + E_{t-2}p_t)] \equiv \beta\pi_0\varepsilon_t + 0.5\beta\pi_1\varepsilon_{t-1}$$

$$y_t = \beta\left(\frac{1}{1+\beta}\right)\varepsilon_t + 0.5\beta\left(\frac{\mu\beta}{(1+\beta)(1+0.5\beta)}\right)\varepsilon_{t-1} \qquad 24.5$$

2) In order to compute the optimal value of μ take variance of y_t:

$$Var(y_t) = E(y_t - E_{t-1}y_t)^2 = E\begin{bmatrix} \beta\left(\frac{1}{1+\beta}\right)\varepsilon_t + 0.5\beta \\ \times\left(\frac{\mu\beta}{(1+\beta)(1+0.5\beta)}\right)\varepsilon_{t-1} \end{bmatrix}^2$$

where the cross products (covariance matrix) are zero as we assume no serial correlation and no heteroscedasticity. Taking the expectations operator inside yields:

$$Var(y_t) = \sigma^2\left(\frac{\beta}{1+\beta}\right)^2 + \sigma^2\left(\frac{0.5\mu\beta^2}{(1+\beta)(1+0.5\beta)}\right)^2$$

If we set the value of $\mu = 0$ (optimal policy response) then the variance of output reduces to:

$$Var(y_t) = \sigma^2\left(\frac{\beta}{1+\beta}\right)^2 = \beta^2\sigma^2\pi_0$$

3) Note that from equation (24.5) we know that:

$$y_t = \left(\frac{\beta}{1+\beta}\right)\varepsilon_t + \left(\frac{0.5\mu\beta^2}{(1+\beta)(1+0.5\beta)}\right)\varepsilon_{t-1}$$

Similarly from the money supply rule equation (24.3) we know that:

$$\varepsilon_t = m_t - \mu y_{t-1} \qquad 24.6$$

$$\varepsilon_{t-1} = m_{t-1} - \mu y_{t-2} \qquad 24.7$$

Substituting for ε_t and ε_{t-1} in equation (24.5) yields:

$$y_t = \left(\frac{\beta}{1+\beta}\right)(m_t - \mu y_{t-1}) + \left(\frac{0.5\mu\beta^2}{(1+\beta)(1+0.5\beta)}\right)(m_{t-1} - \mu y_{t-2})$$

Thus the solution for y_t in terms of current and lagged m_t and y_t is:

$$y_t = \sigma_1 m_t + \sigma_2 m_{t-1} + \sigma_3 y_{t-1} + \sigma_4 y_{t-2} \qquad 24.8$$

where

$$\sigma_1 = \frac{\beta}{1+\beta}$$

$$\sigma_2 = \frac{0.5\mu\beta^2}{(1+\beta)(1+0.5\beta)}$$

$$\sigma_3 = \frac{-\mu\beta}{1+\beta} = -\mu\sigma_1$$

$$\sigma_4 = \frac{-0.5\mu^2\beta^2}{(1+\beta)(1+0.5\beta)} = -\mu\sigma_2$$

4) From equation (24.8) we can calculate optimal money supply rule by setting $y_t = 0$:

$$m_t = \frac{-(\sigma_2 m_{t-1} + \sigma_3 y_{t-1} + \sigma_4 y_{t-2})}{\sigma_1} \qquad 24.9$$

According to our relationship in equation (24.5), this feedback rule for money supply ought to deliver zero fluctuations in output. Thus our expression for money supply from (24.9) after substituting for $\sigma_i's$ give:

$$m_t = \frac{-0.5\mu\beta}{(1+0.5\beta)} m_{t-1} + \mu y_{t-1} + \frac{0.5\mu^2\beta}{(1+0.5\beta)} y_{t-2} + \varepsilon_t$$

Using the lag operartor L we can simplify the above expression as follows:

$$m_t = \mu y_{t-1} + \frac{\varepsilon_t}{\left[1 + \left(\frac{0.5\mu\beta}{1+0.5\beta}\right)L\right]} \qquad 24.10$$

This money supply rule ought to deliver zero fluctuations in output.

5) Substituting equations (24.1) and (24.10) in (24.2) yields the reduced form solution for output (with a new money supply rule) in terms of current and lagged errors. Thus the reduced form equation can be written as:

$$\mu\beta(p_{t-1} - 0.5(E_{t-2}p_{t-1} + E_{t-3}p_{t-1})) +$$

$$\mu\beta a\begin{pmatrix} p_{t-2} - \\ 0.5(E_{t-3}p_{t-2} + E_{t-4}p_{t-2}) \end{pmatrix} + \varepsilon_t =$$

$$p_t + \beta[p_t - 0.5(E_{t-1}p_t + E_{t-2}p_t)] + ap_{t-1} +$$
$$a\beta\begin{bmatrix} p_{t-1} \\ -0.5(E_{t-2}p_{t-1} + E_{t-3}p_{t-1}) \end{bmatrix}$$

where

$$a = \frac{0.5\mu\beta}{1+0.5\beta}$$

Using the Muth method we can write the solution for p_t as:

$$p_t = \bar{p} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}$$

Ignoring the constants and expanding results in :

$$\begin{aligned}
p_t &= \pi_0\varepsilon_t + \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \\
E_{t-1}p_t &= \pi_1\varepsilon_{t-1} + \pi_2\varepsilon_{t-2} + ... \\
E_{t-2}p_t &= \pi_2\varepsilon_{t-2} + \pi_3\varepsilon_{t-3} + ... \\
p_{t-1} &= \pi_0\varepsilon_{t-1} + \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... \\
E_{t-2}p_{t-1} &= \pi_1\varepsilon_{t-2} + \pi_2\varepsilon_{t-3} + ... \\
E_{t-3}p_{t-1} &= \pi_2\varepsilon_{t-3} + \pi_3\varepsilon_{t-4} + ... \\
p_{t-2} &= \pi_0\varepsilon_{t-2} + \pi_1\varepsilon_{t-3} + \pi_2\varepsilon_{t-4} + ... \\
E_{t-3}p_{t-2} &= \pi_1\varepsilon_{t-3} + \pi_2\varepsilon_{t-4} + ... \\
E_{t-4}p_{t-2} &= \pi_2\varepsilon_{t-4} + \pi_3\varepsilon_{t-5} + ...
\end{aligned}$$

Plugging in these values in the reduced form equation and collecting terms in ε_t, ε_{t-1}, ε_{t-2}......

$$\varepsilon_t : \ 1 = \pi_0 + \beta\pi_0$$

$$\therefore \ \pi_0 = \left(\frac{1}{1+\beta} \right)$$

$$\varepsilon_{t-1} : \mu\beta\pi_0 = \pi_1 + 0.5\beta\pi_1 + a\pi_0 + a\beta\pi_0$$

$$\therefore \ \pi_1 = \frac{\mu\beta - a - a\beta}{(1+\beta)(1+0.5\beta)}$$

The identities in the other errors are irrelevant for computing the solution for output.

Substituting the solution for a, π_0 and π_1 in the New Keynesian Phillips curve yields solution for output in terms of current and past ε_t:

$$y_t = \beta\left[(1+\beta)^{-1}\varepsilon_t + 0.25\mu\beta\left((1+\beta)(1+0.5\beta)^2\right)^{-1}\varepsilon_{t-1} \right]$$

Note that the feedback rule for money supply $m_t = \mu y_{t-1} + \frac{\varepsilon_t}{\left[1+\left(\frac{0.5\mu\beta}{1+0.5\beta}\right)L\right]}$ ought to have delivered zero fluctuations in output. However, it has delivered fluctuations nearly as large as the feedback rule. It was Lucas (1976) who first pointed out that if expectations are formed rationally, then unless the estimated model equations are genuinely structural, the implications drawn from such models may be seriously flawed. Where we went wrong was that we used the reduced form to calculate the optimal rule. But that reduced form itself depends on the monetary rule because the rule influences (rational) expectations and hence behaviour of agents.