# 4. Time-inconsistency and Optimal Monetary Policy

Tutorial 25

Suppose the government's objective function is given by $u = -\left(\frac{1}{2}\right)\pi_t^2 + a(y_t - y^\circ)$, where $\pi$ is the inflation rate (with the inflation target being 0); $y$ is the log of output and $y^\circ$ is the target for output.

The government can, by setting money supply, choose $\pi_t$ subject to the Phillip curve: $y_t = y^* + b(\pi_t - \pi_t^e)$, (with $y^*$ being 0).

The private sector decides $\pi_t^e$ *before* t; the government sets $\pi_t$ *during* t.

(a) Suppose the government announces $\pi_t$ *before* t and binds itself (by some external penalty, e.g. set and delivered by the IMF) to stick to it. What can you say about its optimal $\pi_t$?

What do you think will be actual $\pi_t$ , $y_t$?

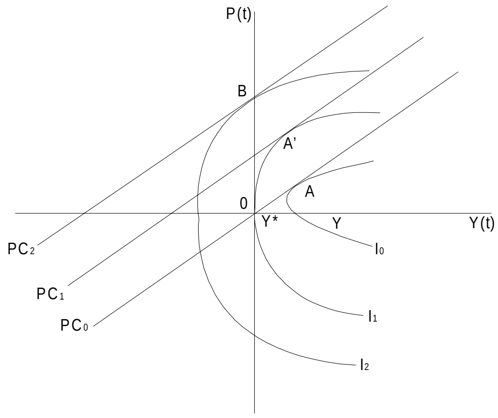
(b) Suppose now, that it cannot bind itself. What do you think will happen then?

## Solution

(a) If the government announces $\pi_t$ before t and is bound by some external penalty to stick to it, then we have the 'pre-commitment' outcome (point 0 in the figure) where $\pi_t = 0, \pi_t^e = 0$. The government's utility function will then be

$$u = -\left(\frac{1}{2}\right)(0)^2 + a[b(0-0) - y^\circ] \equiv ay^\circ$$

which will be less than its utility for 'cheating outcome' (point A) for which $\pi_t \succ 0, \pi_t^e = 0$. This is because it represents higher utility than the pre-commitment outcome. Note that the pre-commitment outcome ensures that $y_t = y^* = 0$, while $y_t \succ 0$ when the government cheats.




(b) If the government cannot bind itself, then the discretionary outcome (point B) will result for which $\pi_t = \pi_t^e \neq 0$.

This can be seen as follows:

The government's optimising exercise is to max $u = -\left(\frac{1}{2}\right)\pi_t^2 + a(y_t - y^\circ)$ subject to $y_t = y^* + b(\pi_t - \pi_t^e)$. The first order condition gives $\pi_t = ab$ from above.On the other hand the private sector's utility function is something like $u = -\left(\frac{1}{2}\right)(\pi_t - \pi_t^e)^2$ which gives $\pi_t^e = \pi_t = ab$. It follows that $y_t = y^* = 0$. Note that government's utility is much lower (at point B) now (when it cannot bind itself) than the pre-commitment (point 0) outcome. This is because the government's utility is $u = -\left(\frac{1}{2}\right)(ab)^2 + a(0 - y^\circ) \equiv -\left(\frac{1}{2}\right)a^2b^2 - ay^\circ \prec -ay^\circ$ (the pre-commitment outcome).

The following illustration shows that under Rational Expectations there are prima facie reasons to believe that in general stabilisation policy, even if effective, will not improve welfare. The main exception being where distortions (benefit/tax system) are positively correlated with the cycle.

# Tutorial 26

Let the social welfare function be quadratic in both output and inflation. The parameter $\alpha$ reflects the relative importance of output and inflation in the social welfare function.

$$L = \frac{1}{2}(y_t - y^*)^2 + \frac{1}{2}\alpha(\pi_t - \pi^*)^2, \qquad y^* \succ \bar{y}, \ \ \alpha \succ 0$$

Subject to the Phillips curve:

$$y_t = \bar{y} + \beta(\pi_t - \pi_t^e), \qquad \beta \succ 0$$

where $y_t$ is the log of output, $\bar{y}$ is the natural rate/potential output and $\pi^*$ is the inflation target.

(a) Suppose the government announces $\pi_t$ *before* t and binds itself to stick to it. What can you say about its optimal $\pi_t$? What do you think will be actual $\pi_t$ , $y_t$?
(b) Suppose now, that it cannot bind itself. What do you think will happen then?

## Solution

a) Substituting the Phillips curve into the quadratic loss function yields:

$$L = \frac{1}{2}(\bar{y} + \beta(\pi_t - \pi_t^e) - y^*)^2 + \frac{1}{2}\alpha(\pi_t - \pi^*)^2 \qquad 26.1$$

Differentiate (26.1) w.r.t $\pi_t$ in order to get optimal inflation:

$$\Rightarrow 0 = \frac{\partial L}{\partial \pi_t} = 2\left(\frac{1}{2}\right)(\bar{y} + \beta(\pi_t - \pi_t^e) - y^*)\beta + 2\left(\frac{1}{2}\right)\alpha(\pi_t - \pi^*)$$
$$\Rightarrow 0 = (\bar{y} + \beta(\pi_t - \pi_t^e) - y^*)\beta + \alpha(\pi_t - \pi^*)$$
$$\Rightarrow (\beta^2 + \alpha)\pi_t = \beta^2\pi_t^e + \alpha\pi^* - \beta(\bar{y} - y^*)$$

or

$$\pi_t = \frac{\beta^2\pi_t^e + \alpha\pi^* + \beta(y^* - \bar{y})}{(\beta^2 + \alpha)} \qquad 26.2$$

Note that $\pi_t^e$ enters the solution for $\pi_t$, so we have to find $\pi_t^e$ first. In order to find expected inflation run the expectations operator along equation (26.2).

$$\Rightarrow \pi_t^e = \frac{\beta^2\pi_t^e + \alpha\pi^* + \beta(y^* - \bar{y})}{(\beta^2 + \alpha)}$$
$$\Rightarrow \pi_t^e\left(1 - \frac{\beta^2}{\beta^2 + \alpha}\right) = \frac{\alpha\pi^* + \beta(y^* - \bar{y})}{(\beta^2 + \alpha)}$$

$$\pi_t^e = \pi^* + \frac{\beta}{\alpha}(y^* - \bar{y}) \qquad 26.3$$

Substituting the expression for expected inflation (26.3) in (26.2) yields solution for optimal inflation:

$$\Rightarrow \pi_t = \frac{\beta^2\left[\pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})\right] + \alpha\pi^* + \beta(y^* - \bar{y})}{(\beta^2 + \alpha)}$$
$$\Rightarrow \pi_t = \frac{(\beta^2 + \alpha)\pi^* + \beta(y^* - \bar{y})\left(\frac{\beta^2}{\alpha} + 1\right)}{(\beta^2 + \alpha)}$$

$$\therefore \pi_t = \pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})$$

If the government can deliver an inflation of $\left[\pi_t = \pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})\right]$ and people expect inflation to be $\pi^*$, then $y_t \succ \bar{y}$ (its preferable outcome). Since, policy makers make a binding commitment before expected inflation is determined, expected inflation would turn out to be equal to actual inflation. Substituting $\pi_t = \pi_t^e$ in the Phillips curve yields:

$y_t = \bar{y}$ (output equals its natural rate)

Since the commitment is binding the social welfare function becomes:

$$L = \underset{\pi}{Min} = \frac{1}{2}(\bar{y} + \beta(\pi_t - \pi_t^e) - y^*)^2 + \frac{1}{2}\alpha(\pi_t - \pi^*)^2$$

or

$$L = \underset{\pi}{Min} = \frac{1}{2}(\bar{y} - y^*)^2 + \frac{1}{2}\alpha(\pi_t - \pi^*)^2$$

Differentiate this loss function w.r.t $\pi_t$ yields solution for actual inflation:

$$\Rightarrow 0 = \frac{\partial L}{\partial \pi_t} = 2\left(\frac{1}{2}\right)\alpha(\pi_t - \pi^*)$$

$$\therefore \pi_t = \pi^*$$

Note that actual inflation equals authorities target for inflation when the commitment is binding. Thus, transparency and credibility ensures a *Pareto optimal* outcome for the society.

b) When authorities cannot bind themself to their commitment then actual and expected inflation are given by:

$$\pi_t = \pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})$$
$$\pi_t^e = \pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})$$

Substituting $\pi_t = \pi_t^e$ in the Phillips curve yields:

$y_t = \bar{y}$ (output equals its natural rate)

Note that although $y_t = \bar{y}$, actual inflation $\left[\pi^* + \frac{\beta}{\alpha}(y^* - \bar{y})\right]$ is greater than $\pi^*$ when authorities cannot bind themselves. The outcome is *Pareto sub-optimal* and provides a theoretical justification for *rules* rather than *discretion*. All that the policymaker's discretion does is to increase inflation with out affecting output.

# Tutorial 27

Let the loss function minimised by the authorities be quadratic in both output and inflation. The parameter $\lambda$ reflects the relative importance of output and inflation in the loss function.

$$L = \frac{E\left[\pi_t^2 + \lambda(y_t - \bar{y})^2\right]}{2}, \qquad \lambda \succ 0, \qquad \bar{y} \succ 0 \qquad 27.1$$

where $E$ is the expectations operator and $\bar{y}$ denotes target level of output. The target level of output exceeds the natural rate i.e., $\bar{y} \succ y^*$.

Let the Phillips curve be given as:

$$y_t = (\pi_t - \pi_t^e) - \varepsilon_t \qquad 27.2$$

where $\varepsilon_t$ is a shock observed by the central bank when setting policy but not by the private sector when setting wages; $\varepsilon$ is symmetrically distributed, with mean zero and constant variance.

The policy rule is given by:

$$\pi_t(\varepsilon) = \bar{m} + m_t \varepsilon_t \qquad 27.3$$

where $\bar{m}$ and $m$ are optimal choice of coefficients in two different policy environments i.e., one in which the policy maker commits to a choice of $\bar{m}$ and $m$ in advance or say the policy is chosen under discretion.

(a) Suppose the government can commit itself to a policy rule of the form (27.3), what is the optimal choice of $\bar{m}$ and $m$? Compute the equilibrium level of output and inflation as well?

(b) Suppose now, that it cannot bind itself. What do you think will happen then?

## Solution

a) Since ε is random and is not observed by the private sector, from (27.3) we know that the private sector would form their expectations by setting $\pi_t^e = \bar{m}$. In order to compute the optimal (government) policy rule set $\pi_t^e = \bar{m}$ into equation (27.2) and substitute the resulting expression into the loss function:

$$\Rightarrow L = \frac{E\left[\pi_t^2 + \lambda(\pi_t - \bar{m} - \varepsilon_t - \bar{y})^2\right]}{2}$$

Substituting for $\pi_t$ policy rule (27.3) into the loss function yields:

$$\Rightarrow L = \frac{E\left[(\bar{m} + m_t\varepsilon_t)^2 + \lambda\{(\bar{m} + m_t\varepsilon_t) - \bar{m} - \varepsilon_t - \bar{y}\}^2\right]}{2}$$

$$L = \frac{E\left[\bar{m}^2 + 2\bar{m}m_t\varepsilon_t + m_t^2\varepsilon_t^2 + \lambda\left((m_t\varepsilon_t - \varepsilon_t)^2 + \bar{y}^2 - 2(m_t\varepsilon_t - \varepsilon_t)m_t\right)\right]}{2}$$

Running the expectations operator along the loss function yields:

$$L = \frac{(\bar{m})^2 + m_t^2\sigma^2 + \lambda\left[m_t^2\sigma^2 + \sigma^2 - 2m_t\sigma^2 + \bar{y}^2\right]}{2}$$

Differentiating this expression w.r.t $\bar{m}$ and $m$ yields optimal policy choice under binding commitment:

$$0 = \frac{\partial L}{\partial \bar{m}} = \frac{2\bar{m}}{2} \Rightarrow \bar{m} = 0$$

$$0 = \frac{\partial L}{\partial m_t} = \frac{1}{2}[2m_t\sigma^2 + \lambda(2m_t\sigma^2 - 2\sigma^2)] \Rightarrow m_t = \frac{\lambda}{1+\lambda}$$

From the first order conditions w.r.t these parameters, we obtain the equilibrium state-contingent policy rules with commitment. Substituting these equilibrium conditions in

(27.3) gives expression for actual inflation:

$$\pi_t(\varepsilon) = \bar{m} + m_t\varepsilon_t$$

Note that $\bar{m} = 0$. Thus:

$$\pi_t(\varepsilon) = \left(\frac{\lambda}{1+\lambda}\right)\varepsilon_t \qquad 27.4$$

Substituting for actual and expected inflation in equation (27.2) yields solution for output:

$$y_t = (\pi_t - \pi_t^e) - \varepsilon_t \equiv \left(\left(\frac{\lambda}{1+\lambda}\right)\varepsilon_t - \bar{m}\right) - \varepsilon_t$$

Since $\bar{m} = 0$ we can simplify the solution for output further:

$$y_t = \left(\frac{\lambda}{1+\lambda}\right)\varepsilon_t - \varepsilon_t \equiv -\left(\frac{1}{1+\lambda}\right)\varepsilon_t$$

b) Consider now the more realistic discretionary monetary regime, where it is impossible for the authorities to commit in advance. The equilibrium policy rule can be obtained by substituting (27.3) in (27.2):

$$y_t = (\bar{m} + m_t\varepsilon_t - \pi_t^e) - \varepsilon_t \qquad 27.5$$

Substituting (27.3) and (27.5) in the loss function yields:

$$\Rightarrow \; L = \frac{E\left[(\bar{m} + m_t\varepsilon_t)^2 + \lambda((\bar{m} + m_t\varepsilon_t - \pi_t^e) - \varepsilon_t - \bar{y})^2\right]}{2}$$

Differentiating this expression w.r.t $\bar{m}$ and $m$ yields:

$$0 = \frac{\partial L}{\partial \bar{m}} = \frac{E[2(\bar{m} + m_t\varepsilon_t) + 2\lambda((\bar{m} + m_t\varepsilon_t - \pi_t^e) - \varepsilon_t - \bar{y})]}{2}$$

$$0 = \frac{\partial L}{\partial m_t} = \frac{E[2(\bar{m} + m_t\varepsilon_t)\varepsilon_t + 2\lambda((\bar{m} + m_t\varepsilon_t - \pi_t^e) - \varepsilon_t - \bar{y})\varepsilon_t]}{2}$$

It follows that equilibrium policy rule must satisfy:

$$\bar{m} + m_t\varepsilon_t = -\lambda(\bar{m} + m_t\varepsilon_t - \pi_t^e - \varepsilon_t - \bar{y})$$

Since we know that $\pi_t(\varepsilon) = \bar{m} + m_t\varepsilon_t$, substituting above yields:

$$\pi_t(\varepsilon) + \lambda(\pi_t - \pi_t^e - \varepsilon_t - \bar{y}) = 0 \qquad 27.6$$

Note that in the solution for actual inflation (under discretion) $\pi_t^e$ appears. In order to find $\pi_t^e$ run the expectations operator along equation (27.6):

$$\pi_t^e = \lambda\ \bar{y} \qquad 27.7$$

Substituting (27.7) in (27.6) yields solution for actual inflation:

$$\pi_t = \lambda\ \bar{y} + \left(\frac{1}{1+\lambda}\right)\varepsilon_t \qquad 27.8$$

Substituting for actual and expected inflation in equation (27.2) yields solution for output which turns out to be identical to the pre-commitment outcome:

$$y_t = (\pi_t - \pi_t^e) - \varepsilon_t \equiv \left(\frac{1}{1+\lambda}\right)\varepsilon_t - \varepsilon_t \equiv -\left(\frac{1}{1+\lambda}\right)\varepsilon_t$$

If we compare the discretionary outcome with the pre-commitment outcome, we see that the inability to commit results in a higher inflation, but leaves output unaffected $\lambda\ \bar{y} + \left(\frac{1}{1+\lambda}\right)\varepsilon_t \succ \left(\frac{\lambda}{1+\lambda}\right)\varepsilon_t$; where $\lambda\ \bar{y}$ denotes the inflationary bias under discretion.

# Tutorial 28

**Independent Central Bank**:

Suppose society's preferences are represented by the following utility function:

$u^s = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y_t - \bar{y})^2\right]$, where symbols have their usual meaning.

Here $y_t$ is given by: $y_t = y^* + c(p_t - p_t^e) + u_t$ (where $u_t$ is a 'white noise' error).

A *tough* central banker (TCB) with a bias towards price stability, has an objective function:

$$u^{TCB} = -\left(\frac{1}{2}\right)p_t^2$$

A *weak* central banker (WCB) has the same preferences as society, i.e.

$$u^{WCB} = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y_t - \bar{y})^2\right]$$

(a) Find the expected utility of society given that if faces a *tough* central banker.

(b) Find the expected utility of society given that if faces a *weak* central banker.

(c) Comment on the economic implications of (a) and (b).

## Solution

a) The *tough* central banker's preference is given by $u^{TCB} = -\left(\frac{1}{2}\right)p_t^2$.
Differentiating w.r.t $p_t$ yields solution for actual and expected price level under a tough central banker.

$$0 = \frac{\partial u^{TCB}}{\partial p_t} = p_t$$

$\therefore\ p_t^e = 0$ (because agents' know that a (TCB) would deliver zero inflation)

Substituting $p_t = p_t^e = 0$ in the Phillips curve yields:

$$y_t = y^* + c(p_t - p_t^e) + u_t \equiv y^* + u_t$$

Substituting $y_t = y^* + u_t$ into the society's preference function yields:

$$\Rightarrow u^s = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y^* + u_t - \bar{y})^2\right]$$
$$\Rightarrow u^s = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y^* - \bar{y})^2 + a(u_t)^2 + 2a(y^* - \bar{y})u_t\right]$$

Running the expectations operator along the utility function yields:

$$E(u^s) = -\left(\frac{1}{2}\right)\left[a(y^* - \bar{y})^2 + a(\sigma)^2\right] \qquad 28.1$$

where $a(y^* - \bar{y})^2$ is the inflation bias and $a(\sigma)^2$ is a measure of instability.

b) The *weak* central banker's preference is given by

$u^{WCB} = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y_t - \bar{y})^2\right]$.

Substituting the Phillips curve in the utility function and the differentiating w.r.t $p_t$ yields solution for actual price level under a weak central banker.

$$\Rightarrow u^{WCB} = -\left(\frac{1}{2}\right)\left[p_t^2 + a(y^* + c(p_t - p_t^e) + u_t - \bar{y})^2\right]$$

$$0 = \frac{\partial u^{WCB}}{\partial p_t} = -p_t - a(y^* + c(p_t - p_t^e) + u_t - \bar{y})c$$

or

$$p_t = ac(\bar{y} + c(p_t^e - p_t) - u_t - y^*)$$

$$\Rightarrow p_t = ac\,\bar{y} + ac^2 p_t^e - ac^2 p_t - acu_t - acy^*$$
$$\Rightarrow p_t(1 + ac^2) = ac(\bar{y} - y^*) + ac^2 p_t^e - acu_t$$

$$p_t = \frac{ac(\bar{y} - y^*) - acu_t}{(1 + ac^2)} + \frac{ac^2}{(1 + ac^2)} p_t^e \qquad 28.2$$

In order to get actual price level we need to know expected price level. Taking expectations of equation (28.2) yields solution for $p_t^e$:

$$p_t^e = \frac{ac(\bar{y} - y^*)}{(1 + ac^2)} + \frac{ac^2}{(1 + ac^2)} p_t^e$$

or

$$p_t^e = ac(\bar{y} - y^*) \qquad 28.3$$

Substituting (28.3) in (28.2) yields solution for actual price level under a weak central banker:

$$p_t = \frac{ac(\bar{y} - y^*) - acu_t}{(1 + ac^2)} + \frac{ac^2}{(1 + ac^2)} ac(\bar{y} - y^*)$$

or

$$p_t = ac(\bar{y} - y^*) - \frac{ac}{(1 + ac^2)} u_t \qquad 28.4$$

Note that from (28.3) and (28.4) we get $p_t - p_t^e = -\frac{ac}{(1+ac^2)} u_t$. Substituting this in the Phillips curve yields solution for output:

$$y_t = y^* + c\left(-\frac{ac}{(1+ac^2)} u_t\right) + u_t$$

Substituting the solution for output and price level into the society's preference function yields:

$$\Rightarrow u^s = -\left(\frac{1}{2}\right)\left[\begin{array}{c} \left(ac(\bar{y} - y^*) - \frac{ac}{(1+ac^2)} u_t\right)^2 + \\ a\left(y^* + c\left(-\frac{ac}{(1+ac^2)} u_t\right) + u_t - \bar{y}\right)^2 \end{array}\right]$$

which reduces to the following expression:

$$u^s = -\left(\frac{1}{2}\right)\left[\begin{array}{c} a^2c^2(\bar{y} - y^*)^2 + \frac{a^2c^2}{(1+ac^2)^2} u_t^2 - \frac{2a^2c^2(\bar{y}-y^*)}{(1+ac^2)} u_t + \\ a((y^* - \bar{y}) + \frac{1}{(1+ac^2)} u_t)^2 \end{array}\right]$$

$$\therefore \; E(u^s) = -\left(\frac{1}{2}\right)\left[(a^2c^2 + a)(\bar{y} - y^*)^2 + \frac{a}{(1+ac^2)} \sigma^2\right]$$

where $(a^2c^2 + a)(\bar{y} - y^*)^2$ is the inflation bias and $\frac{a}{(1+ac^2)}\sigma^2$ is a measure of instability.

| Outcome | Weak Central Bank | | Tough Central bank |
|---|---|---|---|
| Inflation | $(a^2c^2 + a)(\bar{y} - y^*)^2$ | $\succ$ | $a(y^* - \bar{y})^2$ |
| Instability | $\frac{a}{(1+ac^2)}\sigma^2$ | $\prec$ | $a(\sigma)^2$ |

Given that society's preferences are the same as that of a weak central bank i.e., that society desires less instability than could be delivered by a tough central banker, it would be ideal to have a 'COMPROMISE' central banker with a utility function of the form:

$$u^{CCB} = -\left(\frac{1}{2}\right)\left[p_t^2 + \beta(y_t - \bar{y})^2\right], \quad \text{where } 0 \prec \beta \prec a.$$

# Tutorial 29

The central banker's objective function is given by:

$$V = \frac{1}{2}\left[\lambda(y_t - y^* - \chi)^2 + (\pi_t - \pi^*)^2 + \beta(\pi_t - \pi^T)^2\right] \qquad 29.1$$

The central bank's objective function involves output and inflation. In addition, it is penalised for deviation of actual inflation from a target level ($\pi^T$). $\pi^*$ denotes socially optimal inflation rate (which may differ from zero). The parameter $\beta \succ 0$ is the weight placed on deviation from the target inflation rate. The key aspect of this loss function is the parameter $\chi$. The central bank wants to stabilise output around $y^* + \chi$, which exceeds the economy's equilibrium output ( $y^*$) by the constant $\chi$.

In addition we have a Phillips curve of the form:

$$y_t = y^* + \alpha(\pi_t - \pi_t^e) + \varepsilon_t \qquad 29.2$$

where $\varepsilon_t$ denotes a random error. And finally a long-run relationship that,

$$\pi_t = \Delta m_t \qquad 29.3$$

1. Compute actual inflation?

2. What happens when the target rate of inflation is equal to the socially optimal rate?

3. What can you infer when $\beta = 0$?

## Solution

1) Substitute equations (29.3) and (29.2) in (29.1) to get:

$$\Rightarrow V = \frac{1}{2}\left[\begin{array}{c} \lambda(y^* + \alpha(\Delta m_t - \pi_t^e) + \varepsilon_t - y^* - \chi)^2 + (\Delta m_t - \pi^*)^2 \\ +\beta(\Delta m_t - \pi^T)^2 \end{array}\right]$$

Differentiating w.r.t $\Delta m_t$ yields:

$$\Rightarrow 0 = \frac{\partial V}{\partial \Delta m_t} = \lambda\alpha(\alpha(\Delta m_t - \pi_t^e) + \varepsilon_t - \chi) +$$
$$(\Delta m_t - \pi^*) + \beta(\Delta m_t - \pi^T)$$
$$\Rightarrow \alpha^2\lambda(\Delta m_t - \pi_t^e) + \alpha\lambda(\varepsilon_t - \chi) = -(\Delta m_t - \pi^*) - \beta(\Delta m_t - \pi^T)$$

Collecting terms in $\Delta m_t$ yields:

$$\Rightarrow \Delta m_t(\alpha^2\lambda + 1 + \beta) = \pi^* + \beta\pi^T - \alpha\lambda(\varepsilon_t - \chi) + \alpha^2\lambda\pi_t^e$$

or

$$\Delta m_t = \frac{1}{(\alpha^2\lambda + 1 + \beta)}[\pi^* + \beta\pi^T - \alpha\lambda(\varepsilon_t - \chi) + \alpha^2\lambda\pi_t^e] \qquad 29.4$$

We have to compute $\pi_t^e$ for which we take expectations of (29.4):

$$\Delta m_t^e = \frac{1}{(\alpha^2\lambda + 1 + \beta)}[\pi^* + \beta\pi^T + \alpha^2\lambda\pi_t^e + \alpha\lambda\chi] \qquad 29.5$$

Note that

$$\pi_t = \Delta m_t$$

$$\therefore \ \pi_t^e = \Delta m_t^e$$

Substituting $\pi_t^e$ for $\Delta m_t^e$ in (29.5) yields:

$$\pi_t^e = \frac{1}{(\alpha^2\lambda + 1 + \beta)}[\pi^* + \beta\pi^T + \alpha^2\lambda\pi_t^e + \alpha\lambda\chi]$$

Collecting terms in $\pi_t^e$ yields:

$$\pi_t^e = \frac{1}{1+\beta}(\pi^* + \beta\pi^T + \alpha\lambda\chi)$$

Substituting the solution for $\pi_t^e$ in (29.4) yields solution for actual inflation:

$$\Rightarrow \Delta m_t = \frac{1}{(\alpha^2\lambda + 1 + \beta)}\left[\begin{array}{c} \pi^* + \beta\pi^T - \alpha\lambda(\varepsilon_t - \chi) + \frac{\alpha^2\lambda}{1+\beta} \\ \times(\pi^* + \beta\pi^T + \alpha\lambda\chi) \end{array}\right]$$

$$\Rightarrow \Delta m_t = \frac{1}{(\alpha^2\lambda + 1 + \beta)}\left[\begin{array}{c} \pi^*\left(1 + \frac{\alpha^2\lambda}{1+\beta}\right) + \beta\pi^T\left(1 + \frac{\alpha^2\lambda}{1+\beta}\right) - \\ \alpha\lambda\varepsilon_t + \alpha\lambda\chi\left(1 + \frac{\alpha^2\lambda}{1+\beta}\right) \end{array}\right]$$

$$\Rightarrow \Delta m_t = \frac{\pi^*}{1+\beta} + \frac{\beta\pi^T}{1+\beta} + \frac{\alpha\lambda\chi}{1+\beta} - \frac{\alpha\lambda\varepsilon_t}{\alpha^2\lambda + 1 + \beta}$$

or

$$\Delta m_t = \pi^* + \frac{\alpha\lambda\chi}{1+\beta} + \frac{\beta(\pi^T - \pi^*)}{1+\beta} -$$

$$\frac{\alpha\lambda\varepsilon_t}{\alpha^2\lambda + 1 + \beta} \quad \text{(remember that } \pi_t = \Delta m_t) \qquad 29.6$$

2) If the target rate of inflation equals the socially optimal rate then (29.6) reduces to :

$$\Delta m_t = \pi^* + \frac{\alpha\lambda\chi}{1+\beta} - \frac{\alpha\lambda\varepsilon_t}{\alpha^2\lambda + 1 + \beta} \qquad 29.7$$

3) Setting $\beta = 0$ the discretionary outcome with out targeting is:

$$\Delta m_t = \pi^* + \alpha\lambda\chi - \frac{\alpha\lambda\varepsilon_t}{\alpha^2\lambda + 1} \qquad 29.8$$

Comparing these two equations reveals that the targeting penalty reduces the inflation bias.

# Tutorial 30

Consider a simplified version of Barro and Gorden (1983) model. The loss function minimised by the authorities is given by:

$$L = \alpha\pi_t^2 + (y_t - ky^*)^2, \qquad k \succ 1$$

The loss in utility (L) is assumed to be positively related to the square of the inflation rate and the square of the excess demand over the target level. $y^*$ is potential output, $k \succ 1$ reflects the assumption that the target output is above the natural rate. $\alpha$ represents the weight that the authorities attach to inflation.

The Phillips curve is given as:

$$y_t = y^* + \beta(\pi_t - \pi_t^e)$$

1. Compute the equilibrium rate of inflation and the value of the loss function under discretion?

2. What would be the value of the loss function if authorities followed a simple money supply rule which aimed at zero inflation? Compare and contrast the value of the loss function under the two regimes.

## Solution

1) Substituting the aggregate supply curve into the loss function yields:

$$\Rightarrow L = \alpha\pi_t^2 + (y^* + \beta(\pi_t - \pi_t^e) - ky^*)^2$$

Differentiating the loss function w.r.t $\pi_t$ gives the *optimising* discretionary rate of inflation:

$$\Rightarrow 0 = \frac{\partial L}{\partial \pi_t} = \alpha\pi_t + (y^* + \beta(\pi_t - \pi_t^e) - ky^*)\beta$$
$$\Rightarrow -\alpha\pi_t = \beta y^*(1-k) + \beta^2(\pi_t - \pi_t^e)$$

or

$$\pi_t = \frac{\beta[y^*(k-1) + \beta\pi_t^e]}{\alpha + \beta^2} \qquad 30.1$$

Taking expectations of equation (30.1) yields solution for $\pi_t^e$:

$$\Rightarrow \pi_t^e = \frac{\beta[y^*(k-1) + \beta\pi_t^e]}{\alpha + \beta^2}$$

$$\pi_t^e = \frac{\beta}{\alpha}[(k-1)y^*] \qquad 30.2$$

Substituting (30.2) in (30.1) yields solution for actual price level under discretion:

$$\Rightarrow \pi_t = \frac{\beta\left[y^*(k-1) + \beta\left(\frac{\beta}{\alpha}[(k-1)y^*]\right)\right]}{\alpha + \beta^2}$$
$$\Rightarrow \pi_t = \frac{\beta[y^*(k-1)]\left[1 + \frac{\beta^2}{\alpha}\right]}{\alpha + \beta^2}$$

$$\pi_t = \frac{\beta}{\alpha}(k-1)y^* \qquad 30.3$$

Substituting for $\pi_t$ and $\pi_t^e$ in the loss function yields:

$$\Rightarrow L_D = \alpha\left(\frac{\beta}{\alpha}(k-1)y^*\right)^2 + \left(y^* + \beta\left(\frac{\beta}{\alpha}(k-1)y^* - \frac{\beta}{\alpha}(k-1)y^*\right) - ky^*\right)^2$$

or

$$L_D = \frac{\beta^2}{\alpha}[(k-1)y^*]^2 + y^{*2}(1-k)^2 \qquad 30.4$$

2) If the authorities followed a simple rule (targeting zero inflation) and the rule was a binding commitment then:

$$\pi_t = \pi_t^e = 0$$

Then the aggregate supply curve (the Phillips curve) becomes:

$$y_t = y^*$$

Substituting this in the loss function yields the value of the loss function under a rule:

$$\Rightarrow L_R = \alpha\pi_t^2 + (y_t - ky^*)^2 \equiv \alpha\pi_t^2 + (y^* - ky^*)^2$$

$$L_R = y^{*2}(1-k)^2 \qquad 30.5$$

Note that

$$L_D - L_R = \frac{\beta^2}{\alpha}[(k-1)y^*]^2 \succ 0$$

or

$$L_D - L_R = \alpha\pi_t^2 \succ 0$$

By comparing these two loss functions, it is clear that a discretionary policy is inferior to a zero inflation rule because discretion leaves the economy worse off, having now a positive rate of inflation.

# Tutorial 31

**Optimal Inflation Targets**:

Let the policy authorities loss function be given by:

$$L = (\pi_t - \pi^*)^2 + \lambda(y_t - y^*)^2$$

The Phillips curve is given by:

$$y_t = y^N + \alpha(\pi_t - \pi_t^e) + \varepsilon_t$$

where

$y_t$ = output

$y^N$ = potential output

$\pi_t$ = inflation rate

$\pi_t^e$ = inflation expectations (RE)

$\pi^*$ = inflation target

$y^* \succ y^N$ = output target

$\lambda$ = relative weight on output stabilisation

(1) Compute equilibrium inflation and the variance of output under discretion?

(2) Compute equilibrium inflation and the variance of output if we appoint an independent central banker with less weight on output stabilisation (Rogoff 1985) ie., $\lambda^R \prec \lambda$?

(3) Compute equilibrium inflation and the variance of output if we appoint an independent central banker with lower inflation target (Svensson 1997) ie., $\pi^S \prec \pi^*$? Compare and contrast your results to question (2) and (3)? What policies would you advocate to increase output beyond the natural rate?

## Solution

1) Substituting the aggregate supply curve into the loss function yields:

$$\Rightarrow L = (\pi_t - \pi^*)^2 + \lambda\left(y^N + \alpha(\pi_t - \pi_t^e) + \varepsilon_t - y^*\right)^2$$

Differentiating the loss function w.r.t $\pi_t$ gives the *optimising* discretionary rate of inflation:

$$\Rightarrow 0 = \frac{\partial L}{\partial \pi_t} = (\pi_t - \pi^*) + \lambda\alpha\left(y^N + \alpha(\pi_t - \pi_t^e) + \varepsilon_t - y^*\right)$$
$$\Rightarrow \pi_t = \pi^* - \lambda\alpha\left(y^N + \alpha(\pi_t - \pi_t^e) + \varepsilon_t - y^*\right)$$

$$\pi_t = \frac{\pi^* - \lambda\alpha\left(y^N - y^*\right) + \alpha^2\lambda\pi_t^e - \lambda\alpha\varepsilon_t}{1 + \alpha^2\lambda} \qquad 31.1$$

Running the expectations operator along (31.1) yields solution for $\pi_t^e$:

$$\Rightarrow \pi_t^e = \frac{\pi^* - \lambda\alpha\left(y^N - y^*\right) + \alpha^2\lambda\pi_t^e}{1 + \alpha^2\lambda}$$
$$\Rightarrow \pi_t^e\left(1 - \frac{\alpha^2\lambda}{1 + \alpha^2\lambda}\right) = \frac{\pi^* - \lambda\alpha\left(y^N - y^*\right)}{1 + \alpha^2\lambda}$$

$$\pi_t^e = \pi^* - \lambda\alpha\left(y^N - y^*\right) \qquad 31.2$$

Substituting (31.2) in (31.1) yields solution for actual price level:

$$\Rightarrow \pi_t = \frac{\pi^* - \lambda\alpha\left(y^N - y^*\right) + \alpha^2\lambda\left(\pi^* - \lambda\alpha\left(y^N - y^*\right)\right) - \lambda\alpha\varepsilon_t}{1 + \alpha^2\lambda}$$

$$\Rightarrow \pi_t = \frac{\pi^*(1 + \alpha^2\lambda) - \lambda\alpha\left(y^N - y^*\right)(1 + \alpha^2\lambda) - \lambda\alpha\varepsilon_t}{1 + \alpha^2\lambda}$$

$$\pi_t = \pi^* - \lambda\alpha\left(y^N - y^*\right) - \frac{\lambda\alpha\varepsilon_t}{1 + \alpha^2\lambda} \quad \text{(equilibrium inflation under discretion)} \qquad 31.3$$

Substituting the solutions for $\pi_t$ and $\pi_t^e$ in the aggregate supply function yields solution for output:

$$\Rightarrow y_t = y^N + \alpha\left(\begin{array}{c} \pi^* - \lambda\alpha\left(y^N - y^*\right) - \frac{\lambda\alpha\varepsilon_t}{1+\alpha^2\lambda} - \\ \left(\pi^* - \lambda\alpha\left(y^N - y^*\right)\right) \end{array}\right) + \varepsilon_t$$

$$\Rightarrow y_t = y^N - \frac{\lambda\alpha^2\varepsilon_t}{1 + \alpha^2\lambda} + \varepsilon_t$$

$$y_t = y^N + \left(\frac{1}{1 + \alpha^2\lambda}\right)\varepsilon_t$$

Variance of output is given by:

$$E(y_t - E_{t-1}y_t)^2 = E\left(y^N + \left(\frac{1}{1 + \alpha^2\lambda}\right)\varepsilon_t - y^N\right)^2 \equiv \left(\frac{1}{1 + \alpha^2\lambda}\right)^2\sigma^2$$

2) Suppose an independent central banker is appointed with a loss function of the form:

$$L^R = (\pi_t - \pi^*)^2 + \lambda^R(y_t - y^*)^2, \qquad \lambda^R \prec \lambda$$

Following the same procedure as our answer to question (1) i.e., substituting the aggregate supply curve into the new loss function and differentiating the loss function w.r.t $\pi_t$:

$$\pi_t^e = \pi^* - \lambda^R \alpha\left(y^N - y^*\right) \qquad 31.4$$

The solution for actual inflation turns out to be:

$$\pi_t = \pi^* - \lambda^R \alpha\left(y^N - y^*\right) - \frac{\lambda^R \alpha \varepsilon_t}{1 + \alpha^2 \lambda^R} \qquad \text{(lower inflation bias)} \qquad 31.5$$

Substituting the solutions for $\pi_t$ and $\pi_t^e$ in the aggregate supply function yields solution for output:

$$\Rightarrow y_t = y^N + \alpha\left( \begin{array}{c} \pi^* - \lambda^R \alpha\left(y^N - y^*\right) - \frac{\lambda^R \alpha \varepsilon_t}{1+\alpha^2 \lambda^R} - \\ \left(\pi^* - \lambda^R \alpha\left(y^N - y^*\right)\right) \end{array} \right) + \varepsilon_t$$

$$y_t = y^N + \left(\frac{1}{1 + \alpha^2 \lambda^R}\right)\varepsilon_t$$

However, output variability is higher when less weight is placed on output stabilisation:

$$\left(\frac{1}{1 + \alpha^2 \lambda^R}\right)^2 \sigma^2 \succ \left(\frac{1}{1 + \alpha^2 \lambda}\right)^2 \sigma^2$$

Rogoff's result shows that lower inflation bias comes at the cost of increased employment variability. The *second-best* equilibrium cannot be achieved.

3) Suppose an independent central banker is appointed with a loss function of the form:

$$L^S = (\pi_t - \pi^S)^2 + \lambda(y_t - y^*)^2, \qquad \pi^S \prec \pi^*$$

Following the same procedure as our answer to question (1) i.e., substituting the aggregate supply curve into the new loss function and differentiating the loss function w.r.t $\pi_t$:

$$\pi_t^e = \pi^S - \lambda\alpha\left(y^N - y^*\right) \prec \pi^* - \lambda\alpha\left(y^N - y^*\right) \qquad 31.6$$

The solution for actual inflation turns out to be:

$$\pi_t = \pi^S - \lambda\alpha\left(y^N - y^*\right) - \frac{\lambda\alpha\varepsilon_t}{1+\alpha^2\lambda} \qquad \text{(lower inflation bias)} \qquad 31.7$$

Substituting the solutions for $\pi_t$ and $\pi_t^e$ in the aggregate supply function yields solution for output:

$$\Rightarrow y_t = y^N + \alpha\left(\begin{array}{c}\pi^S - \lambda\alpha\left(y^N - y^*\right) - \frac{\lambda^R\alpha\varepsilon_t}{1+\alpha^2\lambda^R} - \\ \left(\pi^S - \lambda\alpha\left(y^N - y^*\right)\right)\end{array}\right) + \varepsilon_t$$

$$y_t = y^N + \left(\frac{1}{1+\alpha^2\lambda}\right)\varepsilon_t$$

Note that we have lower inflation bias than the first outcome and the same amount of variability in output. Svensson's result shows that lower inflation target results in lower average inflation without any effect on output variability. This is in sharp contrast to Rogoff's result that, lower inflation would lead to increased output variability. Remember that distortions create a short-run benefit from surprise inflation. However, the *first-best* outcome is to remove distortions (tax and benefits) altogether. For example use structural policies to make the labour market more flexible. If this is politically infeasible, then a *second-best* outcome can be achieved by commitment to a policy rule (lower inflation target in this case).