5. Representative Agent Models

Tutorial 32

A representative household lives only for one period and has no initial endowment (wealth). The household's objective function (with no uncertainty) is to maximise:

$$U = \ln C_t + \alpha \ln(1 - L_t)$$

Subject to the following constraint:

$$C_t = w_t L_t$$

where C_t denotes consumption, $(1 - L_t)$ denotes leisure and w_t denotes real wage.

a) Derive the intertemporal Euler equations for consumption and labour supply? What can you infer from the equilibrium conditions about labour supply?

b) Suppose household's budget constraint becomes $C_1 + \frac{1}{1+r} C_2 = w_1 L_1 + \frac{1}{1+r} w_2 L_2$ where r is the real interest rate. Will our above conclusions for labour supply hold if the household's horizon is more than one period?

c) What happens to consumption if the subjective discount factor β equals the market discount factor $\frac{1}{1+r}$?

Solution

a) Setting up the Lagrangian (L) for the household's maximisation problem yields:

$$L = \ln C_t + \alpha \ln(1 - L_t) + \lambda(w_t L_t - C_t)$$

The first order conditions for consumption and labour supply yields:

$$0 = \frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda \Rightarrow \frac{1}{C_t} = \lambda \qquad 32.1$$

$$0 = \frac{\partial L}{\partial L_t} = -\left(\frac{\alpha}{1 - L_t}\right) + \lambda w_t \Rightarrow \frac{\alpha}{1 - L_t} = \lambda w_t \qquad 32.2$$

$$0 = \frac{\partial L}{\partial \lambda} = w_t L_t - C_t \qquad 32.3$$

Substituting (32.1) into (32.2) yields:

$$\frac{\alpha}{1 - L_t} = \frac{1}{C_t} w_t \qquad 35.4$$

Since the budget constraint implies $C_t = w_t L_t$, substituting for C_t in (35.4) yields:

$$\frac{\alpha}{1 - L_t} = \frac{1}{w_t L_t} w_t \Rightarrow \frac{\alpha}{1 - L_t} = \frac{1}{L_t} \qquad 32.5$$

Note from (32.5) that labour supply L_t is independent of the real wage. Intuitively the income and substitution effects of a wage change offset each other leaving labour supply unaffected.

b) Suppose the household's horizon is more than one period then, setting up the Lagrangian (L) for the household's maximisation problem yields:

Note that future consumption must be discounted.

$$L = \ln C_1 + \alpha \ln(1 - L_1) + \beta(\ln C_2 + \alpha \ln(1 - L_2)) + \lambda \begin{bmatrix} w_1 L_1 + \frac{1}{1+r} w_2 L_2 \\ -C_1 - \frac{1}{1+r} C_2 \end{bmatrix}$$

where β is the subjective discount factor. The household's choice variables are C_1, C_2, L_1 and L_2.

The first order conditions for consumption and labour supply yields:

$$0 = \frac{\partial L}{\partial C_1} = \frac{1}{C_1} - \lambda \tag{32.6}$$

$$0 = \frac{\partial L}{\partial C_2} = \frac{\beta}{C_2} - \lambda\left[\frac{1}{1+r}\right] \tag{32.7}$$

$$0 = \frac{\partial L}{\partial L_1} = -\left(\frac{\alpha}{1-L_1}\right) + \lambda w_1 \Rightarrow \frac{\alpha}{1-L_1} = \lambda w_1 \equiv \lambda = \frac{\alpha}{(1-L_1)w_1} \tag{32.8}$$

$$0 = \frac{\partial L}{\partial L_2} = -\left(\frac{\beta\alpha}{1-L_2}\right) + \lambda\left[\frac{1}{1+r}\right]w_2 \Rightarrow \frac{\beta\alpha}{1-L_2} = \lambda\left[\frac{w_2}{1+r}\right] \tag{32.9}$$

$$0 = \frac{\partial L}{\partial \lambda} = w_1 L_1 + \frac{1}{1+r} w_2 L_2 - C_1 - \frac{1}{1+r} C_2 \tag{32.10}$$

Substituting (32.8) in (32.9) for λ yields:

$$\Rightarrow \frac{\beta\alpha}{1-L_2} = \left(\frac{\alpha}{(1-L_1)w_1}\right)\left[\frac{w_2}{1+r}\right]$$

or

$$\frac{(1-L_1)}{(1-L_2)} = \frac{w_2}{w_1}\left(\frac{1}{\beta(1+r)}\right) \tag{32.11}$$

From (32.11) it is clear that relative labour supply in the two periods respond to the relative wage. It also implies that a rise in the interest rate raises first-period (today's) labour supply relative to the second period (future). This effect of the interest rate on labour supply is crucial to employment fluctuations in equilibrium business cycle models. The response of labour supply to the relative wage and the interest rate are known in literature as intertemporal substitution in labour supply (Lucas and Rapping, 1969).

c) An important special case is when the subjective discount factor equals the market discount factor. From (32.6) and (32.7) we get:

$$\frac{1}{C_1} = \lambda$$

$$\frac{\beta}{C_2} = \lambda\left[\frac{1}{1+r}\right]$$

Substituting for λ yields:

$$\Rightarrow \frac{\beta}{C_2} = \frac{1}{C_1}\left[\frac{1}{1+r}\right] \equiv \frac{C_1}{C_2} = \frac{1}{\beta(1+r)}$$

If $\beta = \frac{1}{1+r}$ then:

$$C_1 = C_2$$

Note that the time path of aggregate consumption is flat. This prediction of the model captures the idea that, other things being equal, agents would wish to smooth their consumption i.e., consumption smoothing.

Tutorial 33

Consider the following intertemporal household budget constraint of the form:

$$(1+\phi_t)C_t + \frac{b_t}{1+r_t} + p_t S_t = w_{t-1}(1-\psi_{t-1})N_{t-1} +$$

$$\mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) + \frac{b_{t-1}P_{t-1}}{P_t} + \left(p_t + \frac{d_{t-1}P_{t-1}}{P_t}\right)S_{t-1} \qquad 33.1$$

where μ_t is the unemployment benefit, ψ_t is the tax on labour income, ϕ_t is tax on consumption and $\bar{l}$ is normal amount of leisure. N_t is labour supply, S_t is shares owned by the household, b_t is bonds held by the household, d_t, p_t and P_t are dividends, share price and the general price level respectively. All other notations have their usual meaning.

Note that

$$\frac{b_t}{1+r_t} + p_t S_t \equiv \frac{b_t}{1+r_t} + \frac{S_t}{1+r_t} = \left(\frac{1}{1+r_t}\right)(b_t + S_t) = \beta(WE_t) \qquad 33.2$$

where WE_t is household wealth at time t.

1. Show that the permanent income hypothesis (consumption is the annuity value of current and expected financial and human wealth) holds if we have an intertemporal household budget constraint of this form?

Consider the following intertemporal (government's) budget constraint of the form:

$$G_t + b_{t-1} + \mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) =$$

$$w_{t-1}\psi_{t-1}N_{t-1} + \phi_t C_t + \frac{b_t}{1+r_t} + \frac{M_t - M_{t-1}}{P_t} \qquad 33.3$$

where M_t is money supply.

2. Given the following government's intertemporal budget constraint (by imposing the transversality condition $\lim_{N\to\infty} E_t\beta^N b_{t+N} = 0$) derive an expression for government solvency condition?

Solution

1) Substitute (33.2) in (33.1) to get:

$$\Rightarrow (1+\phi_t)C_t + \beta WE_t = w_{t-1}(1-\psi_{t-1})N_{t-1} +$$

$$\mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) + WE_{t-1} + \left(\frac{d_{t-1}P_{t-1}}{P_t}\right)S_{t-1}$$

We can re-write the budget constraint as follows:

$$WE_{t-1} = (1+\phi_t)C_t + \beta WE_t - w_{t-1}(1-\psi_{t-1})N_{t-1} -$$

$$\mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) - \left(\frac{d_{t-1}P_{t-1}}{P_t}\right)S_{t-1} \qquad 33.4$$

Leading (33.4) by one period yields:

$$WE_t = E_t[(1+\phi_{t+1})C_{t+1} - w_t(1-\psi_t)N_t -$$

$$\mu_t\left((1-N_t) - \bar{l}\right) - \left(\frac{d_t P_t}{P_{t+1}}\right)S_t + \beta WE_{t+1}] \qquad 33.5$$

where E_t is the mathematical expectations operator. Substituting (33.5) into (33.4) for WE_t and re-arranging yields:

$$
\begin{aligned}
WE_{t-1} = {} & (1+\phi_t)C_t - w_{t-1}(1-\psi_{t-1})N_{t-1} \\
& - \mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) - \\
& \left(\frac{d_{t-1}P_{t-1}}{P_t}\right)S_{t-1} \\
& + E_t\beta\left(\begin{array}{c} (1+\phi_{t+1})C_{t+1} - w_t(1-\psi_t)N_t - \\ \mu_t\left((1-N_t)-\bar{l}\right) - \left(\frac{d_tP_t}{P_{t+1}}\right)S_t \end{array}\right) \\
& + E_t\beta^2 WE_{t+1}
\end{aligned}
\qquad 33.6
$$

This leaves a term in WE_{t+1} which can be substituted out by leading equation (33.4) two periods and by continuous forward substitution we have:

$$
\begin{aligned}
E_t\beta^N WE_{t+N} = {} & WE_{t-1} - E_t\sum_{i=0}^{N}\beta^i((1+\phi_{t+i})C_{t+i} - \\
& w_{t-1+i}(1-\psi_{t-1+i})N_{t-1+i} - \mu_{t-1+i}\left((1-N_{t-1+i})-\bar{l}\right) \\
& - \left(\frac{d_{t-1+i}P_{t-1+i}}{P_{t+i}}\right)S_{t-1+i})
\end{aligned}
\qquad 33.7
$$

As the forward substitution continues in the limit:

$$
\lim_{N\to\infty} E_t\beta^N WE_{t+N} = 0 \qquad \text{(Transversality condition/no Ponzi game condition)}
$$

Thus we can express (33.7) as:

$$
E_t\sum_{i=0}^{\infty}\beta^i((1+\phi_{t+i})C_{t+i} = WE_{t-1} +
$$

$$
E_t\sum_{i=0}^{\infty}\beta^i\left(\begin{array}{c} w_{t-1+i}(1-\psi_{t-1+i})N_{t-1+i} + \\ \mu_{t-1+i}\left((1-N_{t-1+i})-\bar{l}\right) + \left(\frac{d_{t-1+i}P_{t-1+i}}{P_{t+i}}\right)S_{t-1+i} \end{array}\right)
$$

Expected present value of consumption = initial wealth +expected present value of income.

Friedman's (1957) permanent income theory of consumption is one of the lasting contributions of the 1950's. The permanent income hypothesis is taken to be the proposition that consumption is the annuity value of current and expected financial (and human) wealth, a proposition that turns out to be a special case of the general theory of intertemporal choice.

2) Note that we can re-write the government budget constraint (33.3) as:

$$b_{t-1} = w_{t-1}\psi_{t-1}N_{t-1} + \phi_t C_t + \frac{b_t}{1+r_t} + \frac{M_t - M_{t-1}}{P_t} - G_t - \mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) \qquad 33.8$$

Leading (33.8) one-period and taking expectations at time t yields:

$$b_t = E_t\left(\begin{array}{c} w_t\psi_t N_t + \phi_{t+1}C_{t+1} + \frac{M_{t+1}-M_t}{P_{t+1}} \\ -G_{t+1} - \mu_t\left((1-N_t) - \bar{l}\right) + \frac{b_{t+1}}{1+r_t} \end{array} \right) \qquad 33.9$$

Substituting (33.9) in (33.8) for b_t yields:

$$b_{t-1} = w_{t-1}\psi_{t-1}N_{t-1} + \phi_t C_t + \frac{M_t - M_{t-1}}{P_t} - G_t - \mu_{t-1}\left((1-N_{t-1}) - \bar{l}\right) + E_t\beta\left(w_t\psi_t N_t + \phi_{t+1}C_{t+1} + \frac{M_{t+1}-M_t}{P_{t+1}} - G_{t+1} - \mu_t\left((1-N_t) - \bar{l}\right)\right) + E_t\beta^2 b_{t+1}..... \qquad 33.10$$

This leaves a term in b_{t+1} which can be substituted out by leading (33.8) two-periods and substituting it in (33.10). By continuous forward substitution we have:

$$E_t \beta^N b_{t+N} = b_{t-1} - E_t \sum_{i=0}^{N} \beta^i \left(\begin{array}{c} w_{t-1+i}\psi_{t-1+i}N_{t-1+i} + \phi_{t+i}C_{t+i} + \frac{M_{t+i}-M_{t-1+i}}{P_{t+i}} \\ -G_{t+i} - \mu_{t-1+i}\left((1-N_{t-1+i}) - \bar{l}\right) \end{array} \right) \qquad 33.11$$

As the forward substitution continues in the limit:

$$\lim_{N\to\infty} E_t \beta^N b_{t+N} = 0 \qquad \text{(Government solvency condition)}$$

Thus we can express (33.11) as:

$$\therefore \; b_{t-1} = E_t \sum_{i=0}^{\infty} \beta^i \left(\begin{array}{c} w_{t-1+i}\psi_{t-1+i}N_{t-1+i} + \phi_{t+i}C_{t+i} + \frac{M_{t+i}-M_{t-1+i}}{P_{t+i}} - G_{t+i} - \\ \mu_{t-1+i}\left((1-N_{t-1+i}) - \bar{l}\right) \end{array} \right)$$

Bonds issued at (t-1) which matures at (t) = expected present value of future government surplus.

The government's solvency or transversality condition can therefore be seen to be the commonsense condition that after *some* date it must generate *primary surpluses*: these allow the debt to be paid off.

Tutorial 34

Consider the following intertemporal utility function of a representative agent:

$$U = \sum_{i=0}^{\infty} \beta^i \ln C_{t+i} + \sum_{i=0}^{\infty} \beta^i \ln\left(\frac{M}{P}\right)_{t+i} \qquad 34.1$$

where C is consumption, $(\frac{M}{P})$ is real balances and $\beta = \frac{1}{1+r}$ where r is the individual's rate of time preference.

The intertemporal budget constraint is given by:

$$0 = (1+r)B_{t-1} - \sum_{i=0}^{\infty}\left(\frac{P_{t+i}(C_{t+i} - y) + (M_{t+i} - M_{t+i-1})}{(1+r)^i}\right) \qquad 34.2$$

where P_{t+i} is the price level, y is fixed real income in each time period, M_t is the stock of money, r is the real interest rate and B_t is the nominal quantity of bonds purchased which matures in period t+1.

1. Show that the intertemporal budget constraint of the form

$P_t y + M_{t-1} + (1+r)B_{t-1} = P_t C_t + M_t + B_t$ can be expressed as (34.2)?

2. If so, derive the individual's demand for money?

Solution

1) Re-write the intertemporal budget constraint as:

$$B_t = (1+r)B_{t-1} - P_t(C_t - y) - (M_t - M_{t-1}) \qquad 34.3$$

Lead equation (34.3) by one-period:

$$B_{t+1} = (1+r)B_t - P_{t+1}(C_{t+1} - y) - (M_{t+1} - M_t) \qquad 34.4$$

$$B_t = \frac{P_{t+1}(C_{t+1} - y) + (M_{t+1} - M_t)}{(1+r)} + \frac{B_{t+1}}{1+r} \qquad 34.5$$

Substituting the solution for B_t in (34.3) and re-arranging yields:

$$\frac{B_{t+1}}{1+r} = (1+r)B_{t-1} - \frac{P_{t+1}(C_{t+1} - y)}{(1+r)} - P_t(C_t - y) - \frac{(M_{t+1} - M_t)}{(1+r)} - (M_t - M_{t-1}) \qquad 34.6$$

This leaves a term in B_{t+1} which can be substituted out by leading equation (34.5) by one period and substituting the resulting expression in (34.6). By continuous forward substitution we have:

$$\frac{B_{t+N}}{(1+r)^N} = (1+r)B_{t-1} - \sum_{i=0}^{N}\left(\frac{P_{t+i}(C_{t+i} - y)}{(1+r)^i} + \frac{(M_{t+i} - M_{t-i-1})}{(1+r)^i} \right) \qquad 34.7$$

As the forward substitution continues in the limit:

$$\lim_{N\to\infty} \frac{B_{t+N}}{(1+r)^N} = 0$$

we can re-write (34.7) as:

$$0 = (1+r)B_{t-1} - \sum_{i=0}^{\infty}\left(\frac{P_{t+i}(C_{t+i} - y) + (M_{t+i} - M_{t+i-1})}{(1+r)^i} \right)$$

2) Setting up the Lagrangian yields:

$$L = \sum_{i=0}^{\infty} \beta^i \ln C_{t+i} + \sum_{i=0}^{\infty} \beta^i \ln\left(\frac{M}{P}\right)_{t+i} + \lambda\left[\begin{array}{c} (1+r)B_{t-1} - \\ \sum_{i=0}^{\infty}\left(\frac{P_{t+i}(C_{t+i}-y)+(M_{t+i}-M_{t+i-1})}{(1+r)^i}\right) \end{array}\right]$$

Differentiating w.r.t consumption and money supply yields:

$$0 = \frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda P_t$$

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$$0 = \frac{\partial L}{\partial M_t} = \frac{1}{M_t} - \lambda + \frac{\lambda}{1+r}$$

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$$0 = \frac{\partial L}{\partial \lambda} = (1+r)B_{t-1} - \sum_{i=0}^{\infty}\left(\frac{P_{t+i}(C_{t+i}-y)+(M_{t+i}-M_{t+i-1})}{(1+r)^i}\right)$$

The optimality conditions for consumption and money supply yields:

$$\Rightarrow \frac{C_t}{M_t} = \frac{\frac{r}{(1+r)}}{P_t}$$

$$\therefore \frac{M_t}{P_t} = C_t\left(\frac{1+r}{r}\right)$$

The expression for individuals demand for money shows real balance as a function of consumption and the real interest rate.

Tutorial 35

Consider the following illustration from Sargent (1987(b)). The government's budget constraint at time t is given by;

$$g_t = \tau_t + \int q(x_{t+1}, x_t) b_{t+1}(x_{t+1}) dx_{t+1} - b_t(x_t) + \frac{M_{t+1} - M_t}{P_t}$$

The variable g_t denotes per capita government expenditure at t. The government finances its expenditure by a stream of lump-sum per capita taxes (τ_t). Let x_t be the state of the economy at t, M_t is money supply and P_t is the price level at time t.

Note that $b_{t+1}(x_{t+1})$ is the amount of (t+1) goods that the government promises at (t) to deliver, provided the economy is in the state x_{t+1} at (t+1), and where $q(x_{t+1}, x_t)$ is the current price of one-period ahead goods at time t, provided the next period's state is x_{t+1}. Assume that the government decides to issue only one-period risk free debt, we have $b_{t+1}(x_{t+1}) = b_{t+1}$ for all x_{t+1} so that:

$$\Rightarrow \int q(x_{t+1}, x_t) b_{t+1}(x_{t+1}) dx_{t+1} = b_{t+1} \int q(x_{t+1}, x_t) dx_{t+1}$$

$$\Rightarrow \int q(x_{t+1}, x_t) b_{t+1}(x_{t+1}) dx_{t+1} = b_{t+1} \int \text{present value} \equiv \frac{b_{t+1}}{1 + r_{t+1}} \equiv \frac{b_{t+1}}{R_{1t}}$$

where dx_{t+1} is the probability distribution function and R_{1t} is the gross return on bonds.

1. Show that solvency is guaranteed i.e., the government budget constraint is met at all times given the following budget constraint?

2. What happens to the solvency condition under fixed money supply rule i.e., $M_t = \bar{M}$?

Solution

1) Note that the budget constraint can be re-written as:

$$\Rightarrow g_t = \tau_t + \frac{b_{t+1}}{R_{1t}} - b_t + \frac{M_{t+1} - M_t}{P_t}$$

or

$$b_t = \tau_t - g_t + \frac{b_{t+1}}{R_{1t}} + \frac{M_{t+1} - M_t}{P_t} \qquad 35.1$$

We can iterate upon (35.1) in order to eliminate future $b(x_{t+j})$:

$$b_{t+1} = \tau_{t+1} - g_{t+1} + \frac{b_{t+2}}{R_{1t+1}} + \frac{M_{t+2} - M_{t+1}}{P_{t+1}}$$
$$b_{t+2} = \tau_{t+2} - g_{t+2} + \frac{b_{t+3}}{R_{1t+2}} + \frac{M_{t+3} - M_{t+2}}{P_{t+2}}$$

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Substituting for b_{t+1} in (35.1) yields:

$$\Rightarrow b_t = \tau_t - g_t + \frac{M_{t+1} - M_t}{P_t} + \frac{\tau_{t+1} - g_{t+1} + \frac{b_{t+2}}{R_{1t+1}} + \frac{M_{t+2} - M_{t+1}}{P_{t+1}}}{R_{1t}}$$
$$\Rightarrow b_t = \tau_t - g_t + \frac{M_{t+1} - M_t}{P_t} + \frac{\tau_{t+1} - g_{t+1}}{R_{1t}} + \frac{b_{t+2}}{R_{2t}} + \frac{M_{t+2} - M_{t+1}}{R_{1t} P_{t+1}}$$

$$b_t = \tau_t - g_t + \frac{\tau_{t+1} - g_{t+1}}{R_{1t}} + \frac{M_{t+1} - M_t}{P_t} + \frac{M_{t+2} - M_{t+1}}{R_{1t} P_{t+1}} + \frac{b_{t+2}}{R_{2t}} \qquad 35.2$$

Substituting for $b_{t+2}, \ldots\ldots. \ b_{t+N}$ in (35.2) yields:

$$b_t = \tau_t - g_t + \sum_{j=1}^{\infty} \frac{\tau_{t+j} - g_{t+j}}{R_{jt}} + \frac{M_{t+1} - M_t}{P_t} + \sum_{j=1}^{\infty} \frac{M_{t+j+1} - M_{t+j}}{R_{jt} P_{t+j}}$$

or

$$g_t + b_t + \sum_{j=1}^{\infty} \frac{g_{t+j}}{R_{jt}} = \tau_t + \sum_{j=1}^{\infty} \frac{\tau_{t+j}}{R_{jt}} + \frac{M_{t+1} - M_t}{P_t} + \sum_{j=1}^{\infty} \frac{M_{t+j+1} - M_{t+j}}{R_{jt} P_{t+j}}$$

The above equality states that the present value of government purchases plus the value in goods at t of any government debt due at t, ($b_t(x_t)$) equals the present value of revenues from the lump sum tax and the inflation tax. This solvency condition also implies that monetary and fiscal policies must be co-ordinated in the sense that, given a process for g_t, and a process for τ_t, M_{t+1} cannot be chosen independently for the above solvency condition to hold.

2) Under the fixed money supply rule, $M_t = \bar{M}$ for all t, note that the present value of seignorage is zero and the government solvency condition collapses to:

$$g_t + b_t + \sum_{j=1}^{\infty} \frac{g_{t+j}}{R_{jt}} = \tau_t + \sum_{j=1}^{\infty} \frac{\tau_{t+j}}{R_{jt}}$$

This equality states that the government budget deficit net of interest payments i.e., primary deficit is zero in present value.

Tutorial 36

Consider a simple prototype Real Business Cycle (RBC) model. For simplicity we abstract from the existence of money and government (see McCallum, 1989(b)). The model economy comprises of a representative consumer who maximises his expected utility in a stochastic environment subject to his budget constraint. The utility function is given by:

$$U = Max E_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(C_t,\ L_t\right) \right], \qquad 0 \prec \beta \prec 1 \qquad 36.1$$

Subject to:

$$\begin{aligned} Y_t &= Z_t f(N_t, K_t) && \text{(production function)} \\ K_{t+1} &= (1-\delta)K_t + I_t && \text{(Law of motion for capital stock)} \\ C_t + I_t &= Y_t && \text{(GNP identity)} \\ N_t + L_t &= 1 && \text{(time normalised to unity)} \end{aligned}$$

where N_t is labour supply, δ is the depreciation rate and L_t is leisure time. All other notations have their usual meaning.

1. Derive market equilibrium conditions or the stochastic analogue of the well known Euler equation w.r.t C_t, L_t, K_t and λ_t by setting-up the Lagrangian?

2. Repeat the above exercise when we specify explicit functional forms for u and f i.e.,

$$\begin{aligned} u\left(C_t,\ L_t\right) &= \theta \log C_t + (1-\theta)\log(1-N_t) \\ Z_t f(N_t, K_t) &= Z_t N_t^{\alpha} K_t^{1-\alpha} \end{aligned}$$

Solution

1) Note that

$$C_t + I_t = Z_t f(N_t, K_t)$$
$$I_t = K_{t+1} - (1-\delta)K_t$$
$$L_t = 1 - N_t$$

Thus, the representative consumer's budget constraint is transformed to:

$$C_t + K_{t+1} - (1-\delta)K_t = Z_t f(N_t, K_t) \qquad 36.2$$

At time t, the representative consumer tries to maximise his utility function (36.1) subject to his resource constraint equation (36.2). Setting up the Lagrangian yields:

$$L = MaxE_0\left[\sum_{t=0}^{\infty}\beta^t u\left(C_t,\ 1-N_t\right)\right] + \lambda_t\left[\begin{array}{c} Z_t f(N_t, K_t) - C_t - K_{t+1} \\ +(1-\delta)K_t \end{array}\right]$$

At time t=0, 1....

$$L = u\left(C_0,\ 1-N_0\right) + E_0\beta u\left(C_1,\ 1-N_1\right) + ... +$$

$$\lambda_0[Z_0 f(N_0, K_0) - C_0 - K_1 + (1-\delta)K_0] +$$
$$\lambda_1[Z_1 f(N_1, K_1) - C_1 - K_2 + (1-\delta)K_1] + ...$$

Market equilibrium for this set up can be characterised by the following set of first order conditions:

$$C_t + K_{t+1} - (1-\delta)K_t = Z_t f(N_t, K_t),$$
$$u'\left(C_t,\ 1-N_t\right) = \lambda_t,$$
$$u'\left(C_t,\ 1-N_t\right) = \lambda_t Z_t f_1(N_t, K_t),$$
$$\lambda_t = E_t\beta\lambda_{t+1}[Z_{t+1} f_2(N_{t+1}, K_{t+1}) + (1-\delta)]$$

The second of the above equations equates the marginal utility of consumption to the shadow price of output. The third equates the marginal disutility of labour to labour's marginal product - the real wage. The final equation equates the marginal product of capital to the shadow price of output.

These equations which are the stochastic analogue of the well known Euler equations which characterises the expected behaviour of the economy, determine the time path of the economy's values of labour, consumption, investments and λ.

2) It is important to appreciate the fact that, although these first-order conditions look simple, there are very few functional form for *u* and *f* that would give us explicit closed-form solutions for K_{t+1}, C_t and N_t. One reasonable trick which has been used extensively in literature is a log-linear specification for *u* and a Cobb-Douglas form for *f*. Thus with a log-linear specification for *u* and a Cobb-Douglas form for *f* the representative consumer's objective function becomes:

$$U = MaxE_0\left[\sum_{t=0}^{\infty}\beta^t(\theta\log C_t + (1-\theta)\log(1-N_t))\right]$$

Subject to:

$$C_t + K_{t+1} - (1-\delta)K_t = Z_t N_t^{\alpha} K_t^{1-\alpha}$$

At time t=0, 1....

$$\begin{aligned} L = {} & \theta\log C_0 + (1-\theta)\log(1-N_0) + \\ & E_0\beta[\theta\log C_1 + (1-\theta)\log(1-N_1)] + ... + \end{aligned}$$

$$\lambda_0[Z_0 N_0^{\alpha} K_0^{1-\alpha} - C_0 - K_1 + (1-\delta)K_0] + \lambda_1\left[\begin{array}{c} Z_1 N_1^{\alpha} K_1^{1-\alpha} - C_1 \\ -K_2 + (1-\delta)K_1 \end{array}\right] + ...$$

Maximising w.r.t λ_t, K_{t+1}, C_t and N_t yield:

$$
\begin{aligned}
C_t + K_{t+1} - (1-\delta)K_t &= Z_t N_t^{\alpha} K_t^{1-\alpha}, \\
\frac{\theta}{C_t} &= \lambda_t, \\
\frac{1-\theta}{1-N_t} &= \alpha \lambda_t Z_t N_t^{\alpha-1} K_t^{1-\alpha}, \\
\lambda_t &= (1-\alpha) E_t \beta \lambda_{t+1} [Z_{t+1} N_{t+1}^{\alpha} K_{t+1}^{-\alpha} + (1-\delta)]
\end{aligned}
$$

RBC models place much more emphasis than did the previous equilibrium-approach literature on mechanisms involving cycle propagation. In particular, the primary driving force is taken to be shocks to technology (Z_t), rather than the monetary and fiscal policy disturbances that were emphasised in the earlier equilibrium-approach writings.

Two positions can usefully identified with the RBC school. The weaker of the two is that technology shocks are quantitatively more important than monetary/fiscal disturbances as initiators of business cycle. The stronger position is that monetary/fiscal disturbances are of negligible importance. Recent literature goes further and notes that the propagation mechanism stressed by RBC analysis could be relevant and important even in non-equilibrium models that feature nominal wage and/or price stickiness.

Tutorial 37

Consider the following monetary economy with two markets in each period t; goods for money at the price P_t and labour for money at the wage W_t. There are two representative agents: a firm and a household. The representative household maximises the expected value of discounted future utilities:

$$U = \sum_{t=0}^{\infty} \beta^t \left[\log C_t + \log \frac{M_t}{P_t} + L_t \right] \qquad 37.1$$

Subject to:

$$C_t + \frac{M_t}{P_t} + I_t = \frac{W_t}{P_t} N_t + r_t I_{t-1} + \frac{M_{t-1}}{P_t} \qquad 37.2$$

where N_t is labour supply and the rest of the notations have their usual meaning.

The representative firm maximises profit given a Cobb-Douglas technology:

$$\pi = Y_t - r_t K_t - \frac{W_t}{P_t} N_t \qquad 37.3$$

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha} \qquad 37.4$$

The Law of motion for capital stock is given by:

$$K_{t+1} = (1-\delta) K_t + I_t$$

We assume for simplicity that there is 100% capital depreciation with in a single period i.e., the equation for capital stock thus becomes:

$$K_{t+1} = I_t$$

1. Derive the optimality conditions for the consumer and firm?

2. Derive an expression for C_t, I_t, N_t and Y_t?

Hint: a) Make use of the Law of Iterated Expectations.
b) Use the transversality condition.

3) Can authorities stabilise output in this model?

Solution

1) The representative firm maximises profit subject to its technology constraint. Substituting (37.4) in (37.3) yields:

$$\Rightarrow \ \pi = Z_t N_t^{1-\alpha} K_t^{\alpha} - r_t K_t - \frac{W_t}{P_t} N_t$$

Differentiating the profit function w.r.t N_t and K_t yields:

$$0 = \frac{\partial \pi}{\partial N_t} = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha} - \frac{W_t}{P_t} \qquad 37.5$$

$$0 = \frac{\partial \pi}{\partial K_t} = \alpha Z_t N_t^{1-\alpha} K_t^{\alpha-1} - r_t \qquad 37.6$$

The household on the other hand maximises the expected discounted utility (37.1) subject to the sequence of budget constraint (37.2). Then the optimality conditions for the consumer's program yields:

$$\frac{1}{C_t} = \lambda_t, \qquad 37.7$$

$$1 - N_t = \lambda_t \frac{W_t}{P_t}, \qquad 37.8$$

$$\lambda_t = \beta E_t(\lambda_{t+1} r_{t+1}), \qquad 37.9$$

$$\lambda_t = \frac{1}{M_t} + \beta E_t\left(\lambda_{t+1} \frac{P_t}{P_{t+1}}\right) \qquad 37.10$$

2) Combining (37.7) and (37.9) and using (37.6) for r_t yields:

$$\Rightarrow \frac{1}{C_t} = \beta E_t\left(\frac{1}{C_{t+1}}\alpha Z_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}\right)$$
$$\Rightarrow \frac{1}{C_t} = \beta E_t\left(\frac{1}{C_{t+1}} \times \frac{\alpha Y_{t+1}}{K_{t+1}}\right)$$

Note that the law of motion for capital stock is given by $K_{t+1} = I_t$. Substituting this above yields:

$$\Rightarrow \frac{I_t}{C_t} = \beta E_t\left(\frac{\alpha(C_{t+1} + I_{t+1})}{C_{t+1}}\right)$$
$$\Rightarrow \frac{I_t}{C_t} = \alpha\beta E_t\left(1 + \frac{I_{t+1}}{C_{t+1}}\right)$$

$$\frac{I_t}{C_t} = \alpha\beta + \alpha\beta E_t\left(\frac{I_{t+1}}{C_{t+1}}\right) \qquad 37.11$$

Substituting for $\frac{I_{t+1}}{C_{t+1}}$ and so on i.e., by infinite forward substitution and then imposing the transversality condition and making use of the Law of Iterated Expectations yield solution for C_t:

$$C_t = (1 - \alpha\beta)Y_t \qquad 37.12$$

Making use of the GNP identity $Y_t = C_t + I_t$ yields solution for I_t:

$$I_t = \alpha\beta Y_t \qquad 37.13$$

Combining (37.7) and (37.8) along with (37.5) for real wage yields:

$$\Rightarrow (1 - N_t) = \lambda_t((1-\alpha)Z_t N_t^{-\alpha} K_t^{\alpha})$$
$$\Rightarrow (1 - N_t) = \frac{1}{C_t}\left((1-\alpha)\frac{Y_t}{N_t}\right)$$

Substituting (37.12) for consumption yields solution for labour supply:

$$\Rightarrow (1 - N_t)N_t = \frac{(1-\alpha)}{(1-\alpha\beta)}$$

or

$$N_t = \frac{(1-\alpha)}{(1-\alpha\beta)} \qquad \text{(labour supply is a constant in the solution)} \qquad 37.14$$

Thus, the solution for C_t, I_t, N_t and Y_t can be expressed as:

$$N_t = \bar{N} \qquad \left(\text{where } \bar{N} = \frac{(1-\alpha)}{(1-\alpha\beta)}\right) \qquad 37.15$$

$$Y_t = Z_t K_t^{\alpha} \bar{N}^{1-\alpha} \qquad 37.16$$

$$C_t = (1-\alpha\beta)Y_t \qquad 37.17$$

$$I_t = K_{t+1} = \alpha\beta Y_t \qquad 37.18$$

3) From (37.18) we know that:

$$K_{t+1} = \alpha\beta Y_t$$

$$\therefore K_t = \alpha\beta L Y_t \qquad \text{(where L is the lag operator)}$$

Substituting this in (37.16) yields solution for output in terms of current and past technology shocks:

$$Y_t = Z_t(\alpha\beta L Y_t)^{\alpha} \bar{N}^{1-\alpha}$$

Note that, although the economy is subject to monetary shocks, fluctuations in real variables are driven solely by real shocks. We can see that the introduction of money per se in the utility function in a RBC model does not necessarily give a role for monetary policy. Thus, it is clear that some form of market imperfection is crucial for monetary business cycles. One popular source of propagation mechanism is price rigidity resulting from nominal wage contracts. Modelling wage agreements in this form is justified on the grounds that a relatively large proportion of the manufacturing labour force participate in long-term contracts and hence play a significant role in the transmission of monetary shocks.

Tutorial 38

Lucas cash-in-advance Model:

In this set-up the representative consumer maximises:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

Subject to his budget constraint:

$$C_t + T_t + r_t(x_t)S_t + \frac{1}{P_t}\int l^p_{t+1}(x_{t+1})n\left(x_{t+1}, x_t\right)dx_{t+1}$$
$$= \frac{P_{t-1}d_{t-1}S_{t-1}}{P_t} + r_t(x_t)S_{t-1} + \frac{l^p_t(x_t)}{P_t}$$

where d_t is income (fruit) and x_t is the state of the economy (vector of exogenous variables). Each household (N number of them) have to acquire money in advance in order to carry out transactions. The cash-in-advance constraint is binding only in the case of consumption goods. While one member is shopping for goods the other member is selling it for money, which is taken into the next period. That part of money not needed for shopping is exchanged for income yielding assets (tree S_t (one asset per household)), whose price is given by $r_t(x_t)$. $\frac{l^p_t(x_t)}{P_t}$ is the government's nominal debt and $n\left(x_{t+1}, x_t\right)$ is the price of nominal debt.

Note:

a) Private and government demand for money equals money supply (money market clearing condition).

$$M^p_t + M^g_t = M_{t+1}$$

b) Demand for government bonds by the public is equal to the supply.

$l^p_{t+1}(x_{t+1}) = l_{t+1}(x_{t+1})$

c) Household income has to be carried forward into the next period in the form of money (in order to carry out transactions next period).

$P_t d_t = m_{t+1}$ (where lower case letter denotes per capita values)

d) Government expenditure would have equal its money demand.

$P_t g_t = m^g_t$

e) Household consumption expenditure is lesser than or equal to its money demand as households need more money than their consumption need in order to buy bonds/shares and to pay a lump-sum tax T_t.

$P_t C_t \preceq m^p_t$

f) Since the money and bond market have cleared the goods market by implication clears (Walras's' Law).

$P_t C_t + P_t g_t = P_t d_t$

1. Derive first-order condition w.r.t C_0, C_1, S_0 and $l^p_1(x_1)$?

2. Derive the (i) price of nominal debt (bonds) and (ii) the price of the asset (trees)?

3. Does monetary neutrality hold in this model?

Solution

1) Setting up the Lagrangian at time t=0 for this optimisation problem yields:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) - \lambda_0 \left[\begin{array}{c} C_0 + T_0 + r_0(x_0)S_0 + \frac{1}{P_0} \int l_1^p(x_1) n\left(x_1, x_0\right) dx_1 \\ -\frac{P_{-1}d_{-1}S_{-1}}{P_0} - r_0(x_0)S_{-1} - \frac{l_0^p(x_0)}{P_0} \end{array} \right]$$

$$- \lambda_1 \left[\begin{array}{c} C_1 + T_1 + r_1(x_1)S_1 + \frac{1}{P_1} \int l_2^p(x_2) n\left(x_2, x_1\right) dx_2 \\ -\frac{P_0 d_0 S_0}{P_1} - r_1(x_1)S_0 - \frac{l_1^p(x_1)}{P_1} \end{array} \right]$$

$$0 = \frac{\partial L}{\partial C_0} = u'(C_0) - \lambda_0 \Rightarrow u'(C_0) = \lambda_0 \qquad 38.1$$

$$0 = \frac{\partial L}{\partial C_1} = E_0 \beta u'(C_1) - \lambda_1 \Rightarrow E_0 \beta u'(C_1) = \lambda_1 \qquad 38.2$$

$$0 = \frac{\partial L}{\partial l_1^p(x_1)} = \frac{-\lambda_0}{P_0} n\left(x_1, x_0\right) dx_1 + \frac{\lambda_1}{P_1} \qquad 38.3$$

$$0 = \frac{\partial L}{\partial S_0} = -\lambda_0 r_0(x_0) + \lambda_1 \frac{P_0 d_0}{P_1} + \lambda_1 r_1(x_1) \qquad 38.4$$

2) Substituting (38.1) and (38.2) in (38.3) yields:

$$\Rightarrow \frac{u'(C_0)}{P_0} n\left(x_1, x_0\right) dx_1 = \frac{E_0 \beta u'(C_1)}{P_1}$$

$$\Rightarrow E_0 u'(C_0) n\left(x_1, x_0\right) P_1 = E_0 \beta u'(C_1) P_0$$

$$\Rightarrow n\left(x_1, x_0\right) = E_0 \left[\frac{\beta u'(C_1) P_0}{u'(C_0) P_1} \right]$$

$$\therefore\ n\left(x_{t+1}, x_t\right) = E_0 \left[\frac{\beta u'(C_{t+1}) P_t}{u'(C_t) P_{t+1}} \right]$$

Note that

$$P_t C_t + P_t g_t = P_t d_t \equiv C_t = d_t - g_t$$

Hence we can write the solution for bond price as follows:

$$\Rightarrow n\left(x_{t+1}, x_t\right) = E_0\left[\frac{\beta u'(d_{t+1} - g_{t+1})P_t}{u'(d_t - g_t)P_{t+1}}\right]$$

Note that $P_t d_t = m_{t+1}$. Therefore substituting for P_t yields:

$$\Rightarrow n\left(x_{t+1}, x_t\right) = E_0\left[\frac{\beta u'(d_{t+1} - g_{t+1})\frac{m_{t+1}}{d_t}}{u'(d_t - g_t)\frac{m_{t+2}}{d_{t+1}}}\right]$$

Therefore the price of nominal bond which pays out when x_1 occurs is:

$$\therefore\ n\left(x_{t+1}, x_t\right) = E_0\left[\frac{\beta u'(d_{t+1} - g_{t+1})m_{t+1}d_{t+1}}{u'(d_t - g_t)m_{t+2}d_t}\right]$$

From (38.4) we get:

$$\lambda_0 r_0(x_0) = \lambda_1\left[\frac{P_0 d_0}{P_1} + r_1(x_1)\right] \qquad 38.5$$

Substituting (38.1) and (38.2) in (38.5) yields (remember that ($P_t d_t = m_{t+1}$):

$$\Rightarrow u'(C_0)r_0(x_0) = E_0\beta u'(C_1)\left[\frac{m_1}{P_1} + r_1(x_1)\right]$$

In general we have:

$$u'(C_t)r_t(x_t) = E_0\beta u'(C_{t+1})\left[\frac{m_{t+1}}{P_{t+1}} + r_{t+1}(x_{t+1})\right]$$

or

$$r_t(x_t) = \frac{E_0\beta u'(C_{t+1})}{u'(C_t)}\left[\frac{m_{t+1}}{P_{t+1}} + r_{t+1}(x_{t+1})\right]$$

Note that

$$C_t = d_t - g_t$$

and

$$P_t d_t = m_{t+1}$$
$$P_{t+1} d_{t+1} = m_{t+2} \equiv P_{t+1} = \frac{m_{t+2}}{d_{t+1}}$$

Substituting for C_t and P_{t+1} yields the price of asset:

$$r_t(x_t) = \frac{E_0 \beta u'(d_{t+1} - g_{t+1})}{u'(d_t - g_t)} \left[\frac{m_{t+1} d_{t+1}}{m_{t+2}} + r_{t+1}(x_{t+1}) \right]$$

3) From the price of a real asset (tree) it is clear that asset dividends is in monetary form $\frac{m_{t+1} d_{t+1}}{m_{t+2}}$ and expected future $r_{t+1}(x_{t+1})$. So is the case with government bond yield. In this model a change in the money supply does affect the real value of real assets (trees and government bonds). Hence, monetary policy in this set-up affects real variables mainly because of transaction impediments i.e., return on real assets can be enjoyed only if they are converted into money and spent. This means money is *non-neutral* or in other words is not a *veil*.

Tutorial 39

Consider a simple prototype real business cycle (RBC) model of the form:

$$U = Max E_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} u\big(C_t,\ (1 - N_t)\big) \right] \qquad 39.1$$

Subject to:

$$\begin{aligned} Y_t &= Z_t f(N_t, K_t) \\ K_{t+1} &= (1-\delta)K_t + I_t \\ C_t + I_t &= Y_t \\ Z_{t+1} &= Z_t^{\phi}\varepsilon_{t+1} \end{aligned}$$

where $E_1[\bullet]$ denotes expectations conditional on information at $t = 1$, $0 \prec \beta \prec 1$ is agents' discount factor, C_t denotes consumption, $(1 - N_t)$ is leisure, I_t is investment and $0 \prec \delta \prec 1$ is the depreciation rate of capital.

Note that utility is assumed to be time-separable; that is, the choices of consumption and labour at time t do not affect the marginal utilities of consumption and leisure in any other time period.

1. Derive the equilibrium conditions for the model w.r.t C_t, N_t and K_{t+1} with the following functional forms for u and f:

$$\begin{aligned} u\big(C_t,\ (1-N_t)\big) &= \log C_t + A(1 - N_t) \\ f(N_t, K_t) &= K_t^{\alpha} N_t^{1-\alpha} \end{aligned}$$

2. Set the model (the equilibrium condition that you obtain for question 2) to steady-state and take first-order Taylor series expansion around the steady state values?

Solution

1) Setting up the Lagrangian for this optimisation problem:

$$L = \log C_t + A(1 - N_t) + E_1\beta(\log C_{t+1} + A(1 - N_{t+1})) + \ldots\ldots +$$

$$\lambda_t(Z_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1 - \delta)K_t) +$$

$$\lambda_{t+1}\left(\begin{array}{c} Z_{t+1}K_{t+1}^{\alpha}N_{t+1}^{1-\alpha} - C_{t+1} \\ -K_{t+2} + (1 - \delta)K_{t+1} \end{array} \right) + ..$$

$$\begin{aligned}
0 &= \frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda_t, \\
0 &= \frac{\partial L}{\partial N_t} = -A + \lambda_t[(1 - \alpha)Z_t K_t^{\alpha} N_t^{-\alpha}], \\
0 &= \frac{\partial L}{\partial K_{t+1}} = -\lambda_t + \lambda_{t+1}\beta E_t[\alpha Z_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + (1 - \delta)], \\
0 &= \frac{\partial L}{\partial \lambda_t} = Z_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1 - \delta)K_t
\end{aligned}$$

Market equilibrium is characterised by the following sets of equalities:

$$\begin{aligned}
C_t &= \left(\frac{(1 - \alpha)Z_t K_t^{\alpha} N_t^{-\alpha}}{A} \right) \\
C_t^{-1} &= \beta E_t(C_{t+1}^{-1}[\alpha Z_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + (1 - \delta)]) \\
K_{t+1} &= Z_t K_t^{\alpha} N_t^{1-\alpha} + K_t(1 - \delta) - C_t
\end{aligned}$$

2) The steady-state equilibrium for this economy is one in which the technology shock is assumed to be constant: that is, $Z_t = 1$ for all t, and the values of capital, labour and consumption are constant, $K_{t+1} = \bar{K}$, $N_t = \bar{N}$ and $C_t = \bar{C}$ for all t. Imposing these steady-state conditions on the three equilibrium conditions yield:

$$\bar{C} = \left[\frac{(1-\alpha)}{A} \right] \bar{K}^{\alpha} \quad \bar{N}^{-\alpha}$$

$$\beta^{-1} - 1 + \delta = \alpha \bar{K}^{\alpha-1} \quad \bar{N}^{1-\alpha} \equiv \alpha \left(\frac{\bar{Y}}{\bar{K}} \right)$$

$$\delta \bar{K} = \bar{K}^{\alpha} \quad \bar{N}^{1-\alpha} \quad - \bar{C} \equiv \bar{Y} - \bar{C}$$

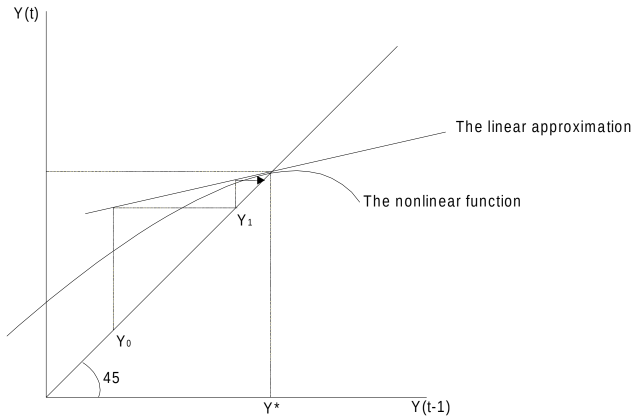
where $\bar{Y}$ denotes the steady state level of output. Given that the model has no analytical solution, the general procedure in the RBC literature is to take a linear approximation (i.e., a first-order Taylor series expansion) of the three equilibrium conditions and the law of motion of the technology shock around the steady-state values $\left(\bar{C}\ ,\ \bar{K}\ ,\ \bar{N}\ ,\ \bar{Z}\right)$. *Provided the stochastic behaviour of the model does not push the economy too far from the steady-state behaviour, the linear approximation will be a good one.*

Short note on linearising a nonlinear model:

Consider the following illustration form Farmer (1993). If we have a nonlinear model of the form:

$$y^{*} = f(y^{*}, x^{*})$$

where f denotes a nonlinear functional form. The idea of linearisation is to recognise that the function $f(\bullet)$ can be approximated by a Taylor series expansion around any point we choose to pick. The stable fixed point of a nonlinear model are good candidates around which to linearise since sequences that begin close to such points will remain close to them. For example, in the figure below, if the initial value of y is set at y_0 when t=0 then y will move to point y_1 at t=1. Successive values of the sequence $[y_t]_{t=0}^{\infty}$ are found the same way. Notice that as the sequence $\left(y_0, y_1, y_2 ...\right)$ approaches y^{*}, the difference between the nonlinear function $f(\bullet)$ and the approximation becomes smaller and smaller.



Using the first-order Taylor series expansion around the steady state ie.,

$$f(x) - f(x^*) = f'(x^*)(x - x^*) + f''(x^*)\frac{(x - x^*)^2}{2!} + f'''(x^*)\frac{(x - x^*)^3}{3!} + \ldots$$

where x is a variable in the function and x^* represents a specific point around which the expansions carried out. From the Taylor series of a function, the linear approximation is obtained by simply dropping all terms of order higher than one.

Thus, for C_t the first-order Taylor series expansion around the steady state would be:

$$\Rightarrow \left(C_t - \bar{C}\right) = \alpha\left[\frac{(1-\alpha)}{A}\right]\bar{K}^{\alpha-1}\ \bar{N}^{-\alpha}\left(K_t - \bar{K}\right) - \alpha\left[\frac{(1-\alpha)}{A}\right]\bar{K}^{\alpha}\ \bar{N}^{-\alpha-1}\left(N_t - \bar{N}\right) + \left[\frac{(1-\alpha)}{A}\right]\bar{K}^{\alpha}\ \bar{N}^{-\alpha}\left(Z_t - \bar{Z}\right)$$

Multiplying and dividing by $\bar{K}$, $\bar{N}$ and $\bar{Z}$ respectively yields:

$$\Rightarrow C_t - \bar{C} = \left[\frac{(1-\alpha)}{A} \right] \bar{K}^{-\alpha} \; \bar{N}^{-\alpha} \left[\begin{array}{c} \alpha\left(\frac{K_t - \bar{K}}{\bar{K}}\right) - \\ \alpha\left(\frac{N_t - \bar{N}}{\bar{N}}\right) + \left(\frac{Z_t - \bar{Z}}{\bar{Z}}\right) \end{array} \right]$$

Expressing consumption as a percentage deviation from steady state and noting that $\bar{C} = \left[\frac{(1-\alpha)}{A} \right] \bar{K}^{\alpha} \; \bar{N}^{-\alpha}$ we get:

$$\tilde{C}_t = \alpha \, \tilde{K}_t - \alpha \, \tilde{N}_t + \tilde{Z}_t$$

where $\tilde{C}_t$ is consumption expressed as a percentage deviation from steady state.

Similarly, using the first-order Taylor series expansion around the steady state for K_{t+1} yields:

$$\left(K_{t+1} - \bar{K}\right) = \alpha \, \bar{K}^{\alpha-1} \; \bar{N}^{1-\alpha} \left(K_t - \bar{K}\right) + (1-\alpha) \, \bar{K}^{\alpha} \; \bar{N}^{-\alpha} \left(N_t - \bar{N}\right) +$$

$$\bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \left(Z_t - \bar{Z}\right) + (1-\delta)\left(K_t - \bar{K}\right) - \left(C_t - \bar{C}\right)$$

Multiplying and dividing by $\bar{K}$, $\bar{N}$, $\bar{C}$ and $\bar{Z}$ respectively yields:

$$\Rightarrow K_{t+1} - \bar{K} = \alpha \, \bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \left(\frac{K_t - \bar{K}}{\bar{K}} \right) + (1-\alpha) \, \bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \left(\frac{N_t - \bar{N}}{\bar{N}} \right) +$$

$$\bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \left(\frac{Z_t - \bar{Z}}{\bar{Z}} \right) + \bar{K} \, (1-\delta)\left(\frac{K_t - \bar{K}}{\bar{K}} \right) - \bar{C} \left(\frac{C_t - \bar{C}}{\bar{C}} \right)$$

or

$$K_{t+1} - \bar{K} = \alpha \, \bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \; \tilde{K}_t + (1-\alpha) \, \bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \; \tilde{N}_t + \bar{K}^{\alpha} \; \bar{N}^{1-\alpha} \; \tilde{Z}_t +$$

$$\bar{K}\ (1-\delta)\ \tilde{K}_t\ -\ \bar{C}\tilde{C}_t$$

Dividing throughout by $\bar{K}$ yields:

$$\tilde{K}_{t+1} = \left[\alpha\ \bar{K}^{\alpha-1}\ \ \bar{N}^{1-\alpha}\ + (1-\delta)\right] \tilde{K}_t\ + (1-\alpha)\ \bar{K}^{\alpha-1}\ \ \bar{N}^{1-\alpha}\ \ \tilde{N}_t\ +$$

$$\bar{K}^{\alpha-1}\ \ \bar{N}^{1-\alpha}\ \ \tilde{Z}_t\ - \left(\frac{\bar{C}}{\bar{K}}\right) \tilde{C}_t$$

Similarly we can express the law of motion for technology in percentage deviation from steady state:

$$\tilde{Z}_{t+1} = \phi\ \tilde{Z}_t\ +\ \tilde{\varepsilon}_{t+1}$$

Tutorial 40

Consider a simple prototype real business cycle (RBC) model of the form:

$$U = MaxE_0\left[\sum_{t=1}^{\infty}\beta^{t-1}\log(C_t)\right]$$

Subject to the following constraints:

$$\begin{aligned} C_t + K_{t+1} &\preceq (1-\delta)K_t + Z_tK_t^{\gamma} \\ Z_t &= Z_{t-1}^{\rho}\varepsilon_t, \qquad\qquad 0 \preceq \rho \preceq 1 \end{aligned}$$

The variable C_t represents consumption and K_t is capital which depreciates at rate δ. The consumer has logarithmic preferences and that he discounts the future at rate $\beta\epsilon\ (0,1)$. The term Z_t represents a technological disturbance to the production technology that is modeled as AR(1) process driven by an innovation ε_t, which is random.

1. Derive the equilibrium conditions (the stochastic analogue of the well known Euler equations) for the model?

2. Set the model to steady-state and take first-order Taylor series expansion of the equilibrium condition?

Solution

1) Setting by the Lagrangian for this optimisation problem:

$$L = \log(C_t) + E_0\log(C_{t+1}) + ... + \lambda_t\left[\begin{array}{c}(1-\delta)K_t + \\ Z_{t-1}^{\rho}\varepsilon_tK_t^{\gamma} - C_t - K_{t+1}\end{array}\right]$$

$$+\lambda_{t+1}\left[(1-\delta)K_{t+1}+Z_t^{\rho}\varepsilon_{t+1}K_{t+1}^{\gamma}-C_{t+1}-K_{t+2}\right]+....$$

$$0=\frac{\partial L}{\partial C_t}=\frac{1}{C_t}-\lambda_t \qquad 40.1$$

$$0=\frac{\partial L}{\partial K_{t+1}}=-\lambda_t+E_0\beta\lambda_{t+1}\left[(1-\delta)+\gamma Z_t^{\rho}\varepsilon_{t+1}K_{t+1}^{\gamma-1}\right] \qquad 40.2$$

Note that $Z_{t+1}=Z_t^{\rho}\varepsilon_{t+1}$. Therefore we can express the above equilibrium condition as:

$$0=\frac{\partial L}{\partial K_{t+1}}=-\lambda_t+E_0\beta\lambda_{t+1}\left[(1-\delta)+\gamma Z_{t+1}K_{t+1}^{\gamma-1}\right]$$

$$0=\frac{\partial L}{\partial \lambda_t}=(1-\delta)K_t+Z_tK_t^{\gamma}-C_t-K_{t+1} \qquad 40.3$$

Market equilibrium is characterised by the following sets of equalities. Substituting (40.2) in (40.1) yields;

$$\Rightarrow \frac{1}{C_t}=\beta E_0\lambda_{t+1}\left[(1-\delta)+\gamma Z_{t+1}K_{t+1}^{\gamma-1}\right]$$

or

$$\frac{1}{C_t}=\beta E_0\frac{1}{C_{t+1}}\left[(1-\delta)+\gamma Z_{t+1}K_{t+1}^{\gamma-1}\right] \qquad 40.4$$

$$K_{t+1}=(1-\delta)K_t+Z_tK_t^{\gamma}-C_t \qquad 40.5$$

2) The steady-state equilibrium for this economy is one in which the values of capital, consumption and technology are constant, $K_t=\bar{K}$, $C_t=\bar{C}$ and $Z_t=\bar{Z}$ for all t. Imposing these steady state condition on the equilibrium conditions yield:

$$\frac{1}{\bar{C}} = \frac{\beta}{\bar{C}}\left((1-\delta) + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\right),$$
$$\bar{K} = (1-\delta)\,\bar{K} + \bar{Z}\bar{K}^{\gamma} - \bar{C}$$

where $\bar{C}$ denotes the steady state level of consumption. Given that the model has no analytical solution the general procedure in the RBC literature is to take a linear approximation (i.e., a first-order Taylor series expansion) of the two equilibrium conditions and the law of motion of the technology shock around the steady-state values $\left(\bar{C},\ \bar{K},\ \bar{Z}\right)$. *Provided the stochastic behaviour of the model does not push the economy too far from the steady-state behaviour, the linear approximation will be a good one*.

Taking the first-order Taylor series approximation around the steady state values $\left(\bar{C},\ \bar{K},\ \bar{Z}\right)$ yield:

$$-\frac{1}{\left(\bar{C}\right)^2}\left(C_t - \bar{C}\right)$$
$$= \beta E_0\left[\begin{array}{c} -\frac{1}{\left(\bar{C}\right)^2}(1-\delta)\left(C_{t+1} - \bar{C}\right) + \gamma\,\bar{Z}\bar{K}^{\gamma-1} - \frac{1}{\left(\bar{C}\right)^2}\left(C_{t+1} - \bar{C}\right) + \\ \frac{1}{\bar{C}}\gamma\,\bar{K}^{\gamma-1}\left(Z_{t+1} - \bar{Z}\right) + \frac{1}{\bar{C}}(\gamma-1)\gamma\,\bar{Z}\bar{K}^{(\gamma-1)-1}\left(K_{t+1} - \bar{K}\right) \end{array}\right]$$

or

$$-\frac{\left(C_t - \bar{C}\right)}{\left(\bar{C}\right)^2} = \beta E_0\left[\begin{array}{c} -\left(\frac{1}{\bar{C}}\right)\widetilde{C_{t+1}}\left((1-\delta) + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\right) + \\ \frac{1}{\bar{C}}\gamma\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{Z_{t+1}} + \frac{1}{\bar{C}}\gamma(\gamma-1)\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{K_{t+1}} \end{array}\right]$$

Multiplying throughout by $\bar{C}$ yields:

$$-\left(\frac{C_t - \bar{C}}{\bar{C}}\right) = \beta E_0\left[\begin{array}{c} -\widetilde{C_{t+1}}\left((1-\delta) + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\right) + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{Z_{t+1}} + \\ \gamma(\gamma-1)\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{K_{t+1}} \end{array}\right]$$

Substituting

$$\frac{1}{\bar{C}} = \frac{\beta}{\bar{C}}\left((1-\delta)+\gamma\,\bar{Z}\bar{K}^{\gamma-1}\right) \text{ above yields:}$$

$$-\widetilde{C_t} = E_0\left[-\widetilde{C_{t+1}} + \beta\gamma\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{Z_{t+1}} + \beta\gamma(\gamma-1)\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{K_{t+1}}\right]$$

Similarly, Taylor series approximation around the steady state for $\bar{K}$ yields:

$$K_{t+1} = (1-\delta)K_t + Z_t K_t^{\gamma} - C_t$$

$$\Rightarrow K_{t+1} - \bar{K} = (1-\delta)\left(K_t - \bar{K}\right) +$$
$$\bar{K}^{\gamma}\left(Z_t - \bar{Z}\right) + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\left(K_t - \bar{K}\right) - \left(C_t - \bar{C}\right)$$

Multiplying and dividing by $\left(\bar{C}\ ,\ \bar{K}\ ,\ \bar{Z}\right)$ yields:

$$\frac{K_{t+1} - \bar{K}}{\bar{K}} = (1-\delta)\widetilde{K_t} + \bar{Z}\bar{K}^{\gamma-1}\,\widetilde{Z_t} + \gamma\,\bar{Z}\bar{K}^{\gamma-1}\,\widetilde{K_t} - \left(\frac{\bar{C}}{\bar{K}}\right)\widetilde{C_t}$$

or

$$\widetilde{K_{t+1}} = \left((1-\delta)+\gamma\,\bar{Z}\bar{K}^{\gamma-1}\right)\widetilde{K_t} + \bar{Z}\bar{K}^{\gamma-1}\,\widetilde{Z_t} - \left(\frac{\bar{C}}{\bar{K}}\right)\widetilde{C_t}$$

Similarly for technology we get:

$$\widetilde{Z_{t+1}} = \rho\widetilde{Z_t} + \widetilde{\varepsilon_{t+1}}$$

Tutorial 41

Assume that a representative agent has a Constant Elasticity of Substitution (CES) utility function of the form:

$$U = Max E_0 \left[\sum_{t=0}^{\infty} \beta^t (\theta C_t^{-\rho} + (1-\theta)(L_t)^{-\rho})^{-\frac{1}{\rho}} \right] \qquad 0 \prec \theta \prec 1$$

where β is the discount factor, C_t is consumption in period t, L_t is the amount of leisure time consumed in period t, and E is the mathematical expectations operator.

The advantage of using this specification is that it does not restrict elasticity of substitution between consumption and leisure to unity. The parameter θ (the distribution parameter) has to do with the relative weights of consumption and leisure in the utility function. The parameter ρ (the substitution parameter) is what determines the value of the (constant) elasticity of substitution. The substitution parameter can take values between minus one and plus infinity i.e., $-1 \prec \rho \prec \infty$.

The representative agent's budget constraint is given by:

$$(1+\phi_t)C_t + \frac{b_t}{1+r_t} + p_t S_t = \frac{P_{t-1} w_{t-1}}{P_t}(1-\psi_{t-1})N_{t-1} +$$

$$\frac{P_{t-1}}{P_t}\mu_{t-1}\left[(1-N_{t-1}) - \bar{l}\right] + \frac{b_{t-1}P_{t-1}}{P_t} + \left(p_t + \frac{d_{t-1}P_{t-1}}{P_t}\right)S_{t-1}$$

where

ϕ_t = indirect taxes
b_t = real value of bonds
r_t = real interest rate
p_t = price of shares
S_t = volume of shares
P_t = general price level
w_t = real wage
ψ_t = income tax
μ_{t-1} = unemployment benefit

$\bar{l}$= normal amount of leisure
d_t = dividends

All other notations have their usual meaning.

1. Derive the optimality conditions for this model?

2. Using the optimality condition, derive a pricing formula for an asset (shares)?

Solution

1) Setting up the Lagrangian and solving for consumption, labour supply, share and bond prices yield:

$$0 = \frac{\partial L}{\partial C_0} = -\frac{1}{\rho}\left[\theta C_0^{-\rho} + (1-\theta)(1-N_0)^{-\rho}\right]^{-\frac{1}{\rho}-1} \bullet$$

$$-\rho\theta C_0^{-\rho-1} - \lambda_0(1+\phi_0)$$

$$0 = \frac{\partial L}{\partial N_0} = -\frac{1}{\rho}\left[\theta C_0^{-\rho} + (1-\theta)(1-N_0)^{-\rho}\right]^{-\frac{1}{\rho}-1} \bullet$$

$$-\rho(1-\theta)(1-N_0)^{-\rho-1}(-1) +$$

$$\lambda_1 E_0\left(\frac{P_0 w_o}{P_1}(1-\psi_0) - \frac{P_0\mu_0}{P_1}\right)$$

$$0 = \frac{\partial L}{\partial b_0} = -\frac{\lambda_0}{(1+r_0)} + E_0\left(\frac{\lambda_1 P_0}{P_1}\right)$$

$$0 = \frac{\partial L}{\partial S_0} = -\lambda_0 p_0 + \lambda_1 E_0\left(p_1 + \frac{d_0 P_0}{P_1}\right)$$

Market equilibrium is characterised by the following set of equalities:

$$-\frac{1}{\rho}\left[\theta C_t^{-\rho} + (1-\theta)(1-N_t)^{-\rho}\right]^{-\frac{1}{\rho}-1} \bullet -\rho\theta C_t^{-\rho-1} = \lambda_t(1+\phi_t) \qquad 41.1$$

$$-\frac{1}{\rho}\left[\theta C_t^{-\rho} + (1-\theta)(1-N_t)^{-\rho}\right]^{-\frac{1}{\rho}-1} \bullet -\rho(1-\theta)(1-N_t)^{-\rho-1} =$$

$$\lambda_{t+1}E_t\left(\frac{P_t w_t}{P_{t+1}}(1-\psi_t)-\frac{P_t\mu_t}{P_{t+1}}\right) \qquad 41.2$$

$$\frac{\lambda_t}{(1+r_t)} = \lambda_{t+1}\left(\frac{P_t}{E_tP_{t+1}}\right) \qquad 41.3$$

$$\lambda_t p_t = \lambda_{t+1}\left(E_t p_{t+1} + \frac{d_t P_t}{E_t P_{t+1}}\right) \qquad 41.4$$

2) Note that we can write (41.3) as:

$$\Rightarrow \frac{\lambda_t}{\lambda_{t+1}} = \frac{P_t(1+r_t)}{E_tP_{t+1}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{E_tP_{t+1}}{P_t(1+r_t)} \qquad 41.5$$

Similarly we can express (41.4) as:

$$\Rightarrow \frac{\lambda_{t+1}}{\lambda_t} = \frac{p_t}{\left(E_t p_{t+1} + \frac{d_t P_t}{E_t P_{t+1}}\right)}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{p_t E_t P_{t+1}}{E_t p_{t+1} E_t P_{t+1} + d_t P_t} \qquad 41.6$$

Equating (41.5) and (41.6) yields solution for share price:

$$\Rightarrow p_t = \frac{1}{P_t(1+r_t)}[E_t p_{t+1} E_t P_{t+1} + d_t P_t]$$

$$\Rightarrow p_t = \frac{E_t p_{t+1} E_t P_{t+1}}{P_t(1+r_t)} + \frac{d_t}{(1+r_t)}$$

$$\Rightarrow p_t = \frac{d_t}{(1+r_t)} + \frac{E_tP_{t+1}}{P_t(1+r_t)}E_t\left[\frac{d_{t+1}}{(1+r_{t+1})} + \frac{E_{t+1}p_{t+2}E_{t+1}P_{t+2}}{E_{t+1}P_{t+1}(1+r_{t+1})}\right]$$

Using the Law of Iterated Expectations ($E_tE_{t+i} = E_t$) yields:

$$p_t = \frac{d_t}{(1+r_t)} + \left(\frac{E_t d_{t+1}}{(1+r_{t+1})}\right)\left(\frac{E_t P_{t+1}}{P_t(1+r_t)}\right) + \left(\frac{E_t P_{t+1}}{P_t(1+r_t)}\right)\left(\frac{E_t P_{t+2}}{E_t P_{t+1}(1+r_{t+1})}\right) +$$

$$E_t\left[\frac{d_{t+2}}{(1+r_{t+2})} + \frac{E_{t+2}p_{t+3}E_{t+2}P_{t+3}}{E_{t+2}P_{t+2}(1+r_{t+2})}\right] \ldots\ldots\ldots\ldots$$

By infinite forward substitution we get:

$$p_t = E_t \sum_{i=0}^{\infty}\left(d_{t+i}^1 \prod_{j=0}^{j=i} q_{t+j}\right)$$

where

$$\begin{aligned} d_{t+i}^1 &= \frac{d_t}{1+r_t} \\ q_{t+j} &= \frac{P_{t+j}}{P_{t+j-1}}(1+r_{t+j-1}), \qquad j \geq 1 \\ q_t &= 1 \end{aligned}$$

Note that stock prices equal expected discounted stream of dividends (Lucas, 1978). Thus stock prices (asset prices - price of trees) varies with whatever the stochastic process driving the value of dividends (fruits).

Tutorial 42

Consider the following (modified) model from McCallum and Nelson (1997). The model economy consists of *n* individuals. In a deterministic setting each household seeks (at time t) to maximise a time-separable utility function of the form: $\sum_{\tau=0}^{\infty} \beta^{\tau} u(C_{t+\tau}, m_{t+\tau})$, where $\beta \epsilon (0,1)$ is the households' discount factor, C_t denotes consumption during t and m_t is the stock of real money balances held at the start of the period.

Households' specialise in production as well. Each produces a single good as restricted by the production function $y_t = f(n_t, k_t)$, where y_t is output, n_t is labour input and k_t is the stock of capital. The functions $f(\bullet)$ and $u(\bullet)$ are assumed to be well behaved i.e., satisfying the *Inada condition*. Note that $\pi_t = \frac{P_{t+1} - P_t}{P_t}$ denotes the inflation rate.

The household's budget constraint is given by:

$$f(n_t, k_t) - T_t = C_t + K_{t+1} - (1-\delta)K_t + w_t(n_t - 1) +$$

$$(1+\pi_t)m_{t+1} - m_t + \frac{b_{t+1}}{1+r_t} - b_t$$

where notations have their usual meaning.

The governments budget constraint is given by:

$$-T_t = (1+\pi_t)m_{t+1} - m_t + \frac{b_{t+1}}{1+r_t} - b_t$$

1. Derive the first-order conditions w.r.t C_t, b_{t+1}, m_{t+1}, π_t and K_{t+1}?

2. Making us of the market equilibrium condition derive an expression for the LM curve?

a) Note that

$$1 + r_t = \frac{1+R_t}{1+\pi_t}$$

b) Assume the following functional form for $u(\bullet)$:

$$u(\bullet) = \theta(1-\sigma)^{-1}C_t^{1-\sigma} + (1-\theta)(1-\eta)^{-1}(m_t)^{1-\eta}$$

3. Derive an expression for the IS curve?

a) Impose the same functional form as above.

b) Upon taking natural logarithms of the resulting equations make use of the following approximation for x small relative to 1.0: $\log(1+x) \simeq x$

Solution

1) Setting up the Lagrangian for this optimisation problem:

$$L = u(C_t, m_t) + \beta u(C_{t+1}, m_{t+1}) + \ldots +$$

$$\lambda_t \begin{bmatrix} f(n_t, k_t) - C_t - K_{t+1} + (1-\delta)K_t - w_t(n_t - 1) \\ -(1+\pi_t)m_{t+1} + m_t - \frac{b_{t+1}}{1+r_t} + b_t + T_t \end{bmatrix}$$

$$+\lambda_{t+1} \begin{bmatrix} f(n_{t+1}, k_{t+1}) - C_{t+1} - K_{t+2} + (1-\delta)K_{t+1} - w_{t+1}(n_{t+1} - 1) \\ -(1+\pi_{t+1})m_{t+2} + m_{t+1} - \frac{b_{t+2}}{1+r_{t+1}} + b_{t+1} + T_{t+1} \end{bmatrix}$$

$$0 = \frac{\partial L}{\partial C_t} = u_1(C_t, m_t) - \lambda_t \qquad 42.1$$

$$0 = \frac{\partial L}{\partial b_{t+1}} = -\frac{\lambda_t}{1 + r_t} + \beta\lambda_{t+1} \qquad 42.2$$

$$0 = \frac{\partial L}{\partial m_{t+1}} = \beta u_2(C_{t+1}, m_{t+1}) - \lambda_t(1 + \pi_t) + \beta\lambda_{t+1} \qquad 42.3$$

$$0 = \frac{\partial L}{\partial n_t} = \lambda_t[f_1(n_t, k_t) - w_t] \qquad 42.4$$

$$0 = \frac{\partial L}{\partial k_{t+1}} = -\lambda_t + \beta\lambda_{t+1}[f_2(n_{t+1}, k_{t+1}) + (1 - \delta)] \qquad 42.5$$

2) Substituting (42.5) for λ_t in (42.3) yields:

$$\Rightarrow \beta u_2(C_{t+1}, m_{t+1}) - \beta\lambda_{t+1}[f_2(n_t, k_t) + (1 - \delta)] \times (1 + \pi_t) + \beta\lambda_{t+1} = 0$$

Substituting (42.1) for λ_{t+1} yields:

$$\beta u_2(C_{t+1}, m_{t+1}) = \beta u_1(C_{t+1}, m_{t+1})[(1 + \pi_t) \times (f_2(n_{t+1}, k_{t+1}) + (1 - \delta)) - 1] \qquad 42.6$$

Substituting (42.5) in (42.2) yields:

$$\Rightarrow \frac{\beta\lambda_{t+1}[f_2(n_{t+1}, k_{t+1}) + (1 - \delta)]}{1 + r_t} = \beta\lambda_{t+1}$$

or

$$r_t = f_2(n_{t+1}, k_{t+1}) - \delta \qquad 42.7$$

Notice that after substituting (42.7) equation (42.6) collapses to:

$$\Rightarrow \beta u_2(C_{t+1}, m_{t+1}) = \beta u_1(C_{t+1}, m_{t+1})[(1 + \pi_t) \times (r_t + \delta + 1 - \delta) - 1]$$

If $1 + r_t = \frac{1+R_t}{1+\pi_t}$ then:

$$\Rightarrow \beta u_2(C_{t+1}, m_{t+1}) = \beta u_1(C_{t+1}, m_{t+1}) R_t$$

$$\frac{u_2(C_{t+1}, m_{t+1})}{u_1(C_{t+1}, m_{t+1})} = R_t \qquad 42.8$$

Under reasonably standard assumption for the functional form equation (42.8) can be solved for as:

$$m_{t+1} = m(C_{t+1}, R_t)$$

If we assume the functional form given above we have;

$$\frac{(1-\theta)}{\theta} \frac{m_{t+1}^{-\eta}}{C_{t+1}^{-\sigma}} = R_t$$

In logarithms we have:

$$\Rightarrow \log M_{t+1} - \log P_{t+1} = -\frac{1}{\eta} \log\left(\frac{\theta}{1-\theta}\right) - \frac{1}{\eta} \log R_t + \frac{\sigma}{\eta} \log C_{t+1}$$

or

$$m_{t+1} = m(C_{t+1}, R_t) \qquad 42.9$$

Thus, we have a relationship expressing end-of-period real money balances as a function of the upcoming period’s consumption spending and the current nominal interest rate. If we assume that C_t provides a satisfactory index of fluctuations in total output (this is a reasonable conjecture because consumption is pro-cyclical), equation (42.9) describes essentially the same type of behaviour as that of a standard LM equation. That is, real money balances are positively related to a transactions variable and negatively related to an opportunity cost variable.

3) Substituting (42.1) in (42.5) yields:

$$\Rightarrow u_1(C_t, m_t) = \beta u_1(C_{t+1}, m_{t+1})[r_t + \delta + 1 - \delta]$$

where we have made use of $f_2(n_{t+1}, k_{t+1}) = r_t + \delta$.

$$\therefore\ u_1(C_t, m_t) = u_1(C_{t+1}, m_{t+1})\beta[1 + r_t]$$

If we impose the functional form given, we have:

$$C_t^{-\sigma} = C_{t+1}^{-\sigma}\,\beta[1 + r_t]$$

Upon taking logarithms we get:

$$\Rightarrow \log C_t = \log C_{t+1} - \frac{1}{\sigma}\log\beta - \frac{1}{\sigma}\log[1 + r_t]$$

or

$$\log C_t = b_0 + b_1 r_t + \log C_{t+1} \qquad 42.10$$

where we have used the approximation $\log(1 + x) \simeq x$ for x small relative to 1.0.

For business cycle purposes we can approximate fluctuations in Y_t with those in C_t because consumption is positively correlated with the cycle. Note that relationship of the form (42.10) is a simplified IS curve with next period's output as an important determinant of the quantity of output demanded in the current period.

Tutorial 43

OLG Model:

A logical extension to a representative agent framework is to assume heterogeneity among agents'. Suppose all generations (young and old) are made up of N identical agents, who's income stream in perishable consumption units is $y - \varepsilon$ when young and ε when old. Consumption when young is C_t and C_{t+1} when old. $0 \prec \varepsilon \prec \frac{y}{2}$ so that they obtain more income when young than at old age. Consumer maximises a logarithmic utility function of the form:

$$MaxU = \ln C_t + \ln C_{t+1}$$

Subject to:

$y - \varepsilon - C_t = l_t$ (young's budget constraint)

$l_t\,(1 + r_t) + \varepsilon = C_{t+1}$ (old's budget constraint)

where l_t represents lending.

1. Assume that there is no investment opportunity other than a loan market and no government in existence. Show that there would be no lending when young and old in this set-up?

2. Suppose the government enters the market and borrows $N\left[\frac{y}{2} - \varepsilon\right]$ to spend in the first period only and pays it off by taxing the T^{th} generation, what happens?

Solution

1) Note that from the old's budget constraint we get:

$$l_t\,(1 + r_t) + \varepsilon = C_{t+1} \;\Rightarrow\; l_t = \frac{C_{t+1} - \varepsilon}{1 + r_t}$$

Substituting l_t in the young's budget constraint we get:

$$\frac{C_{t+1} - \varepsilon}{1 + r_t} = y - \varepsilon - C_t$$

or

$$(y - \varepsilon - C_t)(1 + r_t) + \varepsilon - C_{t+1} = 0 \qquad \text{(budget constraint)}$$

Setting up the Lagrangian for this optimisation problem:

$$L = \ln C_t + \ln C_{t+1} + \lambda[(y - \varepsilon - C_t)(1 + r_t) + \varepsilon - C_{t+1}]$$

Maximising w.r.t C_t, C_{t+1} and λ yields:

$$0 = \frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda(1 + r_t) \Rightarrow \lambda = \frac{1}{C_t(1 + r_t)}$$
$$0 = \frac{\partial L}{\partial C_{t+1}} = \frac{1}{C_{t+1}} - \lambda \Rightarrow \frac{1}{C_{t+1}} = \lambda$$
$$0 = \frac{\partial L}{\partial \lambda} = (y - \varepsilon - C_t)(1 + r_t) + \varepsilon - C_{t+1}$$

From the first two equalities we get:

$$\frac{1}{C_{t+1}} = \frac{1}{C_t(1 + r_t)} \Rightarrow C_{t+1} = C_t(1 + r_t)$$

Substituting for C_{t+1} in the budget constraint yields:

$$\Rightarrow (y - \varepsilon - C_t)(1 + r_t) + \varepsilon - C_t(1 + r_t)$$
$$\Rightarrow (y - \varepsilon)(1 + r_t) + \varepsilon = 2C_t(1 + r_t)$$

$$\therefore\ C_t = \frac{\varepsilon}{2(1 + r_t)} + \frac{(y - \varepsilon)}{2}$$

Note that loans made by each generation is its income (-) consumption ie.,

$$y - \varepsilon - C_t = l_t$$

Substituting for young's consumption yields:

$$y - \varepsilon - \left(\frac{\varepsilon}{2(1 + r_t)} + \frac{(y - \varepsilon)}{2} \right) = l_t$$

or

$$l_t = \frac{y - \varepsilon}{2} - \frac{\varepsilon}{2(1 + r_t)}$$

If we impose market clearing i.e., $\sum_{h=1}^{\infty} l_t = 0$

$$0 = \sum_{h=1}^{\infty} \left(\frac{y - \varepsilon}{2} - \frac{\varepsilon}{2(1 + r_t)} \right)$$

or

$$\Rightarrow (y - \varepsilon)(1 + r_t) = \varepsilon$$

$$(1 + r_t) = \frac{\varepsilon}{y - \varepsilon} \prec 1$$

Negative interest rate would ensure that there is no intergenerational transfer of loans.

Substituting $1 + r_t$ in the solution for consumption yields:

$$\Rightarrow C_t = \frac{(y - \varepsilon)}{2} + \frac{(y - \varepsilon)}{2}$$

or

$C_t = y - \varepsilon$ (consumption when young)

We know that:

$$C_{t+1} = C_t(1 + r_t)$$

Substituting $1 + r_t$ and C_t in the solution for C_{t+1} yields:

$$C_{t+1} = C_t\left(\frac{\varepsilon}{y - \varepsilon}\right) \equiv \varepsilon \qquad \text{(consumption when old)}$$

2) Suppose the government borrows $N\left[\frac{y}{2} - \varepsilon\right]$ then, the young's budget constraint becomes:

$$y - \varepsilon - C_t - \left(\frac{y}{2} - \varepsilon\right) = l_t$$

The old's budget constraint is unaffected.

$$l_t\,(1 + r_t) + \varepsilon = C_{t+1} \;\Rightarrow\; l_t = \frac{C_{t+1} - \varepsilon}{1 + r_t}$$

Substituting l_t in the young's budget constraint we get:

$$\left(\frac{y}{2} - C_t\right)(1 + r_t) + \varepsilon - C_{t+1} = 0$$

Setting up the Lagrangian for this optimisation problem and maximising w.r.t C_t, C_{t+1} and λ yields:

$$\lambda = \frac{1}{C_t(1 + r_t)}$$

$$\lambda = \frac{1}{C_{t+1}}$$

$$\therefore\; C_{t+1} = C_t(1 + r_t)$$

Substituting for C_{t+1} in the budget constraint yields:

$$\Rightarrow \frac{y}{2}(1+r_t)+\varepsilon = 2C_t(1+r_t)$$

$$C_t = \frac{\varepsilon}{2(1+r_t)} + \frac{y}{4}$$

Note that loans made by each generation is its income (-) consumption and imposing market clearing we get:

$$\Rightarrow \frac{y}{2} = \frac{\varepsilon}{(1+r_t)}$$

or

$$(1+r_t) = \frac{\varepsilon}{\frac{y}{2}} \succ \frac{\varepsilon}{y-\varepsilon}$$

Note that government's intervention has raised the interest rate.

Substituting $1 + r_t$ in the solution for consumption yields:

$C_t = \frac{y}{2}$ (Optimal consumption smoothing - consumption while young and old is equal)

Note that government issued liability could perform by intermediating between people who may or may not be willing to lend directly to each other but will be willing to lend to the government which in turn lends it to others. In this case government bonds (debt) are net wealth.

Tutorial 44

Equity Premium Puzzle:

The consumer maximises:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t), \qquad 0 \prec \beta \prec 1 \qquad 44.1$$

Subject to his budget constraint:

$$A_{t+1} = R_t(A_t + Y_t - C_t) \qquad 44.2$$

with A_0 given. Here C_t is consumption at time t, Y_t is labour income, A_t is the amount of a single earning asset valued in the units of the consumption good and R_t is the real gross rate of return on the assets between dates t and (t+1).

1. Show that our equilibrium condition for this optimisation problem is:

$u'(C_t) = \beta E_t[(1 + r_{t+1})u'(C_{t+1})]$
using dynamic programming solution method discussed earlier?

2. With the help of a constant-relative-risk-aversion utility function (CRRA) i.e.,

$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$, show that our Euler equation becomes
$(1 + r_t) = E_t\left((1 + r_t)(1 + g_c)^{-\theta}\right)$.

Note that $g_c = \frac{C_{t+1} - C_t}{C_t}$.

3. What happens if we take second order Taylor series approximation around $r_t = g = 0$?

Assume that $E(r) \bullet E(g) = [E(g)]^2 = 0$ and $\theta cov(r, g) = 0$ (if we have a risk free asset).

Solution

1) At time t=0 the utility function can be expressed as:

$$MaxE_0 = [u(C_0) + \beta u(C_1) + \beta^2 u(C_2) + ...] \quad 44.3$$

or

$$V = Maxu(C_0) + MaxE_0 \beta u(C_1) + ... \quad 44.4$$

Expression (44.4) indicates that the original large optimisation problem on the right side of (44.3) can be broken up into (∞) smaller problems. First, the problem in the innermost bracket is solved. Then the problem on the second innermost bracket is solved and so on sequentially.

Thus we can write:

$$Maxu = u(C_0) + E_0 \beta u(C_1) + ... + \quad 44.5$$

At time t=0 the budget constraint (44.2) becomes:

$$\Rightarrow A_1 = R_0 A_0 + R_0 Y_0 - R_0 C_0$$

or

$$C_0 = A_0 + Y_0 - \frac{A_1}{R_0} \quad 44.6$$

Similarly at time t=1 our budget constraint (44.2) becomes:

$$C_1 = A_1 + Y_1 - \frac{A_2}{R_1} \qquad 44.7$$

Substituting (44.7) in (44.5) yields:

$$Maxu = u(C_0) + E_0\beta u\left(A_1 + Y_1 - \frac{A_2}{R_1}\right) + \ldots\ldots\ldots\ldots + \qquad 44.8$$

Note that at time t=0 we know A_1. From (44.6) we can write A_1 as:

$$A_1 = R_0(Y_0 + A_0 - C_0)$$

Thus (44.8) can be written as:

$$Maxu = u(C_0) + E_0\beta u\left(R_0(Y_0 + A_0 - C_0) + Y_1 - \frac{A_2}{R_1}\right) + \ldots\ldots\ldots\ldots +$$

Maximising w.r.t C_0 yields:

$$0 = \frac{\partial L}{\partial C_0} = u'(C_0) + E_0\beta u'(C_1)(-R_0)$$

In general we have:

$$\Rightarrow u'(C_t) = \beta E_t R_t u'(C_{t+1}) \quad \text{(since } R_t = 1 + r_{t+1}\text{)}$$

$$\therefore u'(C_t) = \beta E_t[(1 + r_{t+1})u'(C_{t+1})]$$

2) Now assume that the representative agent has a CRRA utility function:

$$u(C_t) = \frac{C_t^{1-\theta}}{1 - \theta}$$

or

$$u'(C_t) = \frac{(1-\theta)C_t^{(1-\theta)-1}}{1-\theta} \equiv C_t^{-\theta}$$

With this assumption our Euler equation becomes:

$$\Rightarrow C_t^{-\theta} = \beta E_t[(1+r_{t+1})C_{t+1}^{-\theta}]$$

where θ is the coefficient of relative risk aversion. Dividing both sides by $C_t^{-\theta}$ and multiplying both sides by $(1+r_t)$ yields:

$$\Rightarrow (1+r_t)\frac{C_t^{-\theta}}{C_t^{-\theta}} = \beta E_t\left[\frac{(1+r_{t+1})C_{t+1}^{-\theta}}{C_t^{-\theta}}\right](1+r_t), \quad (\beta = \frac{1}{(1+r_t)})$$

or

$$(1+r_t) = E_t\left[(1+r_{t+1})\frac{C_{t+1}^{-\theta}}{C_t^{-\theta}}\right]$$

Defining growth of consumption as $g_c = \frac{C_{t+1}-C_t}{C_t}$, we can express:

$$\frac{C_{t+1}^{-\theta}}{C_t^{-\theta}} = (1+g_c)^{-\theta}$$

Ignoring the time subscript we get:

$$\therefore\ (1+r) = E_t\left[(1+r)(1+g_c)^{-\theta}\right]$$

3) To see the implications of this equation, we take a second-order Taylor series approximation of the right side of the equation around $r = g = 0$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + ...$$

If $f(r,g) = (1+r)(1+g)^{-\theta}$

Then at time t=0 we have:

$$\Rightarrow f(r_0,g_0) = f'_{r_0}(r_0,g_0)(r-r_0) + f'_{g_0}(r_0,g_0)(g-g_0) + \frac{1}{2}f''_r(r_0,g_0)(r-r_0)^2 +$$

$$\frac{1}{2}f''_g(r_0,g_0)(g-g_0)^2 + \frac{1}{2}f''_{rg}(r_0,g_0)(r-r_0)(g-g_0)$$

Note that

$$\begin{aligned} f'_r &= (1+g)^{-\theta} \\ f''_r &= 0 \\ f'_g &= -\theta(1+r)(1+g)^{-\theta-1} \\ f''_g &= \theta(\theta+1)(1+r)(1+g)^{-\theta-2} \\ f''_{rg} &= \frac{\partial^2 f}{\partial r^2} = (-\theta)(1+g)^{-\theta-1} \end{aligned}$$

Therefore

$$\Rightarrow f(r_0,g_0) = 1 + r + g(-\theta) + \frac{1}{2} \times 0 + \frac{1}{2}\theta(\theta+1)g^2 + (-\theta)rg$$

or

$$f(r_0,g_0) = \left[1 + r - \theta g - \theta rg + \frac{1}{2}\theta(\theta+1)g^2\right]$$

or

$$E(A+B) = E(A) + E(B)$$

$$\Rightarrow E\left[\left[1 + r - \theta g - \theta rg + \frac{1}{2}\theta(\theta+1)g^2\right]\right] = 1 + r$$

$$\Rightarrow E(1) + E(r) - E(\theta g) - \theta E(rg) + \frac{1}{2}\theta(\theta+1)Eg^2 = 1 + r$$

Definition *For any two random variables* $x, y,$ *we have the formula* $E_t xy = E_t x E_t y + cov_t(x,y),$ *where* $cov_t(x,y) \equiv E_t(x - E_t x)(y - E_t y)$. *This formula defines the conditional covariance* $cov_t(x,y)$.

Also note that:

$$E(g^2) = E(g - E(g))^2 + [E(g)]^2 \equiv Var(g) + [E(g)]^2$$

Thus we have:

$$E(r) - \theta E(g) - \theta[E(r) \times E(g) + cov(r,g)] +$$

$$\frac{1}{2}\theta(\theta + 1) \times \left(Var(g) + [E(g)]^2\right) \simeq r$$

When the time period involved is short,

$E(r) \bullet E(g)$ and $[E(g)]^2$ terms are small relative to the others. Omitting these terms and solving the resulting expression for $E(r)$ yields:

$$E(r^M) \simeq r + \theta E(g) + \theta cov(r,g) - \frac{1}{2}\theta(\theta + 1)Var(g) \qquad 44.9$$

If we consider a risk-free asset then (44.9) becomes:

$$r \simeq r + \theta E(g) - \frac{1}{2}\theta(\theta + 1)Var(g) \qquad 44.10$$

Substracting (44.10) from (44.9) yields:

$$E(r^M) - r \simeq \theta cov(r,g)$$

The equation for equity risk premium suggests that excess returns on a portfolio approximately equals the coefficient of risk aversion parameter times the correlation coefficient between consumption growth and the risk free rate.